CS 70 Discrete Mathematics and Probability Theory
Fall 2016 Seshia and Walrand Final

SID: ______________________________

1. TRUE or FALSE?: 2x8=16 points

Clearly put your answers in the answer box on the bottom of this page!

This is what is to be graded. No need to justify!

1. A finite irreducible Markov chain that can go from state 1 to itself in one step is such that $P[X_n = 1]$ must converge as $n \to \infty$.

2. We are given a probability space with sample space Ω and probability $P(\cdot)$. Let $\{A_1, A_2, \ldots, A_5\}$ be pairwise disjoint events whose union is Ω and let B be another event. If $P[B|A_1] = 1$, then $P[A_1|B] \geq P[A_k|B]$ for $k = 1,...,5$.

3. You throw 15 balls independently and uniformly at random into 7 bins. Then, the number of balls in bins 3 and 5 are positively correlated.

4. The number of ways of distributing 10 identical smartphones to Steve, Bill, and Larry is $\binom{10}{3}$.

5. You draw five cards from a perfectly shuffled standard 52-card deck. The probability that the third card is an ace is strictly smaller than the probability that the first card is an ace.

6. If $Q[Y|X] = 3 + 2X$, then $L[Y|X] = 3 + 2X$.

7. Assume that $E[(Y - 3\cos(5X) - 13\cos(7X))\cos(nX))] = 0$ for $n = 0, 1, \ldots, 9$. Then the values of $a_0, a_1, \ldots, a_9$ that minimize $E((Y - a_0 - a_1\cos(X) - a_2\cos(2X) - \cdots - a_9\cos(9X))^2)$ are such that $a_k = 0$ for $k \notin \{5,7\}$ and $a_5 = 3, a_7 = 13$.

8. Let p be a prime number. The set of all functions from $\{0,1,2,..., p-1\}$ to $\mathbb{Q}$ is uncountably infinite.

Answer Box

(*Answer TRUE or FALSE in the corresponding box below.*)

1	2	3	4	5	6	7	8

2. Short Answers: 4x5=20 points
Provide a clear and concise justification of your answer.

1. You roll a six-sided balanced die until you get the first 6. How many dots do you accumulate, on average, including the 6 on the last roll?

2. Alice, Bob, and Charles want to flip a coin to decide who pays for lunch, and want each to have the same probability of paying. However, they only have a biased coin and they don't even know its bias. Explain a mechanism that achieves the objective.

3. You have to pass two courses in sequence, A then B, to be able to declare CS as your major. When you take a course for the first time, you pass it with probability $1/2$. If you take it for the second time, you pass it with probability $2/3$. If you fail a course twice, you cannot declare CS as your major. What is the probability that you can declare CS as your major?

4. You pick n points independently and uniformly at random inside a unit circle. Let Z be the distance of the furthest point to the center. What is $E(Z)$?

5. Suppose we draw cards from a standard 52-card deck without replacement. What is the expected number of cards we must draw before we get all 13 spades (including the last spade)?

3. Rolling Dice I: 3+3+3+3=12 points
Provide a clear and concise justification of your answer.

In this problem, you roll two balanced six-sided dice.

1. What is the probability that the number of dots on the first die is at least twice the number on the second die?

2. What is the probability that the first die yields 4 if the total number of dots on the two dice is 8?

3. What is the probability that the sum is 8 given that the second die yields 4?

4. What is the probability that the number of dots on the first die is at least as large as on the second?

4. Tossing Coins: 6+6= 12 points
Provide a clear and concise justification of your answer.

1. You are given two coins A and B. One is biased with $P(H) = 0.6$ and the second is fair. You don't know which is biased; it is equally likely to be A or B. You flip the pair of coins repeatedly (both each time) until at least one of the two coins yields H. It turns out that you have to flip them 27 times and that coin A then yields H while coin B yields T. What is the probability that A is the biased coin?

2. You flip a fair coin 10 times. Let X be the total number of heads in the first 5 flips and Y the total number of heads in the odd flips, i.e., in flips 1,3,5,7, and 9. (a) Find $E[X|Y]$. (b) Find $E[Y|X]$.

5. Rolling Dice II: 8+6=14 points
Provide a clear and concise justification of your answer.

1. You roll a six-sided balanced die until either you get a 6 or the number of dots on a roll exceeds the number of dots on the previous roll. How many rolls do you need, on average? Write the equations that you have to solve to find the answer. Do not solve the equations.

2. You have a die and want to check if it is balanced. To do this, you decide to test the mean. You roll the die 10^4 times and find an average number of dots per roll equal to 3.2.
(a) Using Chebyshev's inequality and a simple upper bound on the variance, find a 95% confidence interval for the mean. (*Hint: It can be shown that variance is bounded by 36.*)
(b) Are you 95% sure that the die is not balanced?

6. Graphs and Chains: 8+6=14 points
Provide a clear and concise justification of your answer.

1. Imagine you are traversing a 3-dimensional hypercube. Beginning at the vertex 000, you repeatedly randomly pick any outgoing edge from your current vertex and follow it to another vertex. What is the expected number of edges you must traverse before reaching the vertex 111 for the first time?

2. Consider a Markov chain on $\{0,1\}$ with $P(0,1) = \alpha = P(1,0)$ for some $\alpha \in [0,1]$. Let $X_0, X_1, X_2, \ldots, X_n$ be the successive values of the Markov chain.
(i) Propose a method to estimate α.
(ii) Using Chebyshev's inequality, what is a 95% confidence interval for α.

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7. Erdos-Renyi Random Graph: 4+2+4+2=12 points
Provide a clear and concise justification of your answer.

Random graph model is a key idea in network analysis. Erdos-Renyi Graph Model is the most popular and simplest network model. Suppose you have n vertices and you leave them unconnected at the beginning. Then for every two vertices, you draw an undirected edge with probability p. (p is a fixed number for all edges). Suppose we label all the vertices in an order 1, 2, ..., n.

1. Prove that in any E-R graph, if u is a vertex of odd degree in, then there exists a path from u to another vertex v of the graph where v also has odd degree. A *path* is a sequence of edges (v_1, v_2) ,(v_2, v_3), ..., (v_{n_2}, v_{n_1}), (v_{n_1}, v_n), where no vertex is repeated.

2. What's the probability that there are 0 paths of length 2 connecting vertices 1 and 2?

3. Let X_l be the number of paths of length l connecting vertices 1 and 2. What's the expectation and variance of X_2?

4. What's the expectation of X_l? (You should give an expression in terms of n, p and l and you could leave a summation, integral or product if that's easier to write the final answer.)

8. Polling for 2020: 10 points Provide a clear and concise justification of your answer.

We poll 100 men and 200 women. Among those, 30 men and 160 women declare they would vote for Michelle Obama for President in 2020. Assume that, among the voters in the general election, there will be an equal number of men and women and that your poll is representative of the voting in the general election. What is your 95% confidence interval for the fraction of people who will vote for Michelle? Would you advise her to run? Justify your answer.

To solve this problem, we suggest that you follow the following steps: (1) What is your estimate of the fraction of the people who will vote for Michelle in the general election? (2) What is its variance?; (3) What is an upper bound on that variance?; (4) Write down Chebyshev and use the upper bound; (5) Choose the distance from the mean so that the Chebyshev bound is 5%; (6) What is the resulting confidence interval?

9. Random Variables: 3+6+6+8+7=30 points
Provide a clear and concise justification of your answer.

1. Define 'random variable' in one sentence. (No need for a justification.)

2. Let $X_n, n \geq 1$ be i.i.d. with mean μ and variance σ^2. Let also $A_n = (X_1 + \cdots + X_n)/n$. (a) Using Chebyshev, find a 90% confidence interval for μ. (b) Apply this result to the situation where the X_n are 1 when a coin yields H and zero otherwise.

3. You choose a point (X, Y) uniformly at random in the triangle with vertices $(-1,0), (0,1), (1,0)$. (a) Find $E[Y|X]$. (b) Find $L[Y|X]$. (c) Are X and Y positively-, negatively-, or un-correlated?

4. Let Y and Z be independent random variables where Y is Poisson with mean 60 and Z is Poisson with mean 10. (a) Find $L[Y|X]$ where $X = Y + Z$. (b) Find $E[Y|X]$. (*Hint:* Recall that if V is Poisson with mean λ, then $E(V) = \lambda$ and $var(V) = \lambda$. Also, recall that the sum of Poisson independent random variables is Poisson.)

5. Let X be a discrete random variable that takes values in $[0,1]$. Show that its variance is at most $1/4$. *Hint:* Show that the random variable that takes at most n different values in $(0,1)$ and that has the largest variance is Bernoulli $1/2$. To do this, argue by contradiction and assume that X takes a value x in $[0.5,1)$ with probability p. Replace that value by 1 with probability $1/2$ and $2x-1$ with probability $1/2$. Show that the variance increases.