CS 70 Discrete Mathematics and Probability Theory
Fall 2017 Ramchandran and Rao

Final Solutions

1. Discrete Math: True/False (2pts/part,9 parts. 18 points)

1. (True/False) If $d|m$ and $d|n$ then $d|(m-n)$. (Recall $x|y$ means "the integer x divides the integer y".)

   Answer: True. $m = kd$ and $n = zd$ so $m-n = kd - zd = d(k-z)$ so $d|m-n$.

2. (True/False) If $d|mn$ then $d|n$ or $d|m$.

   Answer: False. $6|6$ but $6 \not| 3$ and $6 \not| 2$.

3. (True/False) $\neg P \implies \neg Q$ implies that $\neg Q \implies \neg P$.

   Answer: False. Converses of a statement are not always true.

4. (True/False) Recall a tournament graph on n vertices has an directed edge between every pair of vertices in exactly one direction. If the tournament has a directed cycle of length 5, there is a smaller directed cycle.

   Answer: True. Tournament graphs have a cycle of length 3 if they have a cycle at all.

5. (True/False) Given a 3-colorable n vertex graph, there exists a way to add a vertex and edges to the graph where the new vertex has degree $\lceil 2n/3 \rceil$ and still have a 3-colorable graph.

   Answer: True. At least one color has at most $\lfloor n/3 \rfloor$ vertices in it. One can connect the new vertex to the other two colors.

   Answer: True. Take out a vertex, recurse, now you have d neighbhors so one color must be available.

6. (True/False) If there are two different stable pairings in stable marriage instance, then it cannot be the case that all men have the same preference list.

   Answer: True. Consider the contrapositive. If all men have the same preference list, then the favorite woman must get her favorite, the second favorite woman gets must have her favorite or her second favorite if taken. And so on. So, there must one be one stable pairing.

7. (True/False) In a broken run of the traditional marriage algorithm where exactly one woman accidentally rejects a man but ends up with a man she likes better. The resulting pairing is stable.

   Answer: True. It is still true that any man she rejected is worse than any man who asked her, and any man who didn't ask her is paired with someone he likes better.

8. (True/False) If men ask women in reverse order of preference one still gets a stable pairing, but this time it is female optimal.

   Answer: False. Consider an instance the only stable pairing is one where all people are matched to their favorites and the men's least favorite form a pairing.

9. (True/False) The length of every cycle in the hypercube is even.

   Answer: True. It is bipartite or two-colorable which only can happen if all cycles are even.

2. Discrete Math:Short Answer (3 pts/part, 16 parts. 48 points.

1. $\neg(\forall x, P(x) \vee Q(x)) \equiv \exists x,$_________. (Fill in the blank.) (Make sure the negation is fully distributed.)

   Answer: $\neg P(x) \wedge \neg Q(x)$

2. What is the size of $\{a \pmod n, 2a \pmod n, 3a \pmod n, \ldots, (n-1)a \pmod n\}$ if $gcd(a,n) = 1$?

   Answer: $n-1$ since a is relatively prime to n, a has an inverse and the image is the same size as the pre-image.

3. What is $2^{16} \pmod 7$?

   Answer: 2. Either repeated squaring, or Fermat and look at 2^4

4. What is the size of the set $\{ay \pmod{pq} : y \in \{1,\ldots,pq-1\}\}$ when a is not a multiple of p or q?

   Answer: $pq-1$ since a has an inverse mod pq.

5. What is the size of the set $\{ay \pmod{pq} : y \in \{1,\ldots,pq-1\}\}$ when a is a multiple of p (but not q)?

   Answer: q since there are q multiples of p modulo pq.

6. What is $5^{60} \pmod{77}$?

   Answer: 1. This is $5^{(p-1)(q-1)} \pmod{pq}$ which is 1.

7. What is $7^{60} \pmod{77}$?

   Answer: 56. $7^{60} = 1 \pmod{11}$, and it is a multiple of 7. Enumerating gives 56.

8. What is the minimum number of degree 1 vertices in an n-vertex tree for $n > 1$? (Answer could be an expression that involves n.)

   Answer: 2. Think about a path. Every tree has two degree one vertices since $\sum_v d(v) = 2m = 2n-2$ and if every vertex but one had degree at least 2 the sum would be $2(n-1)+1$ for the vertex of degree 1.

9. What is the maximum number of degree 1 vertices in an n-vertex tree? (Answer could be an expression that involves n.)

   Answer: $n-1$. Think of a star.

10. What is the maximum number of edges in any simple planar graph with 5 vertices?

    Answer: $\binom{5}{2}-1$. One less than the number of edges in K_n.

11. Consider Professor Rao's public key (N,e) and secret key d, which has 512 bits. Professor Rao wants to share the secret key with his three children where any two can recover the secret d.

    Answer: (This is make believe, the truth is Professor Rao's kids know all his non-work passwords.)

    (a) What degree polynomial should he use?
    Answer: Degree 1. Two points recover a line.

    (b) How big should the field over which we are working be? (That is, how big should the modulus be for the modular arithmetic that we use.)
    Answer: The modulus $p > 2^{512}$ to accomodate the secret in the y-intercept.

12. What is an error polynomial for the Berlekamp-Welsh method where the corrupted packets correspond to points with x-values $0,1$ and 3 working modulo 11?

    Answer: $(x-1)(x)(x-3) = x^3-4x^2+3x$

13. For degree (at most) d non-zero distinct polynomials, $P(x)$ and $Q(x)$, what is the maximum number of roots that $P(x)-Q(x)$ can have?

    Answer: d. It is still a non-zero degree at most d polynomial, so it can have at most d roots.

14. We have to assign 750 students to rooms in CS70.

    (a) How many ways are there to do this in 3 rooms of capacity 240, 250, and 260?
    Answer: $\binom{750}{240}\binom{510}{250}$.

    (b) How many ways are there to do this in 3 rooms of capacity 250, 260, and 270? (Notice there will be a total of 30 empty seats in this case.) (An expression that may involve summations.)
    Answer: $\sum_{i=0}^{30}\sum_{j=0}^{30-i}\binom{750}{250-i}\binom{500+i}{260-j}$

3. A Quick Proof. (12 points.)

You have n coins $C_1, C_2, \ldots, C_n$ for $n \in \mathbb{N}$. Each coin is weighted differently so that the probability that coin C_i comes up heads is $\frac{1}{2i+1}$. Prove by induction that if the n coins are tossed, then the probability of getting an odd number of heads is $\frac{n}{2n+1}$.

Answer: Base Case: For one coin the probability of heads (and an odd number of heads) is $1/(2(1)+1) = 1/3 = n/(2n+1)$ for $n = 1$.

Our inductive hypothesis is that getting an odd number of heads for n coins occurs with probability $n/(2n+1)$.

To get an odd number of heads for $n+1$ coins, one either gets an odd number on the first n and gets a tails, which occurs with probability

$$\frac{n}{2n+1} \times (1 - \frac{1}{2n+3}) = \frac{n(2n+2)}{(2n+1)(2n+3)}$$

using the induction hypothesis and the setup, or the first n are even and one gets a heads, which occurs with probability

$$(1 - \frac{n}{2n+1}) \times (\frac{1}{2n+3} = \frac{n+1}{(2n+1)(2n+3)}.$$

Adding them together gives $\frac{(n+1)(2n+1)}{(2n+1)(2n+3)} = \frac{n+1}{2n+3} = \frac{n+1}{2(n+1)+1}$ which is what we desire.

□

1. Base case.
2. State your induction hypothesis.
3. Do the inductive step.

4. Probability:True/False. (2pts/parts,6 parts. 12 points)

1. (True/ False) If false give a counterexample in the space provided next to the true-false bubbles. The example is graded.
   (a) If X, Y are independent, then $cov(X, Y) = 0$.
   Answer: True.
   But, proof is $cov(X,Y) = E[XY] - E[X]E[Y]$, but for independent random variables $E[XY] = E[X]E[Y]$.
   (b) If $cov(X, Y) = 0$, then X, Y are independent.
   Answer: False.
   But example of a joint is (X, Y) are $\{(1, -1), (-1, 1)\}$ with probability $1/3$ each and (X, Y) are $\{(\sqrt{2}, \sqrt{2}), (-\sqrt{2}, -\sqrt{2})\}$ with probability $1/6$ each.
2. (True/False) If X_1 and X_2 are i.i.d. Exp(1) random variables, $cov(\min(X_1, X_2), \max(X_1, X_2)) = 0$.
   Answer: False. Memoryless property suggests that $X_{\max} - X_{\min}$ is independent of $X_{\min}$ not that $X_{\max}$ is independent of $X_{\min}$. In fact, $X_{\max}$ is the sum of two independent random variables so it has a positive covariance with $X_{\min}$.

3. (True or False) If $X \sim Geom(p)$, then $E[X+m \mid X>n] = m+n+E[X]$.

   Answer: True. It is $m+E[X \mid X>n] = m+n+E[X]$ since the geometric distribution is memoryless.

4. The CLT can be used to bound the probability that a random variable is far from its mean.

   Answer: False. The Chebyshev bound is a bound and is always true, the CLT is an approximation that gets better with larger n.

5. Quick conceptual questions. (4 pts part, 3 parts. 12 points.)

1. Explain in words what it means when the covariance between two random variables X and Y is (a) positive (b) negative (c) zero.

   Answer: If the correlation is positive, one generally expects the Y to be above average when X is above average, and if negative then Y would be below average when X is above average, and if zero than any linear function of X does not estimate Y and better than not knowing X, e.g., they could be independent, but other situations also exist.

2. In a sentence or two describe: (i) the difference between Bayesian and non-Bayesian linear regression: (ii) how you can study both perspectives using a single framework.

   Answer: Bayesian is about minimizing the expectation over a distribution where non-bayesian is about minimizing the error on sample points. If one takes the distribution for sample points to be uniform the bayesian derivation formulas apply to the non-Bayesian situation.

3. A sequence of random variables $X_0, X_1, X_2, X_3, \dots$ is a Markov chain if: (fill in the blank)

   Answer: $\Pr[X_{t+1} \mid X_t] = \Pr[X_{t+1} \mid X_t, X_{t-1}, \dots, X_0]$ for all t and $Pr[X_{t'+1} \mid X_{t'}] = Pr[X_{t+1} \mid X_t]$.

6. Probability: Short Answer. 3pts/part. 30 parts. 90 pts.

1. In a class of 24 students, what is the probability that at least two students have the same birthday? (Assume the number of days in a year is 365. Answer is an expression, possibly with products. No need to simplify.)

   Answer: $1 - \Pi_{i=1}^{23} \frac{365-i}{365}$. The latter expression calculates the probability that all students have different birthdays.

2. Two real numbers are chosen uniformly from the unit interval. What is the probability that their sum is less than or equal to 1 given that one of them is less than or equal to 1/2?

   Answer: $2/3$.

   $P[sum < 1]/(1 - Pr[\text{both greater than } 1/2]) = 0.5/0.75 = 2/3$

3. If X, Y are independent continuous-valued random variables uniform in $[0,1]$. What is $E[X \mid X+Y = 1.5]$?

   Answer: .75 by linearity of expectation and symmetry. $E[X+Y \mid X+Y = 1.5] = E[X \mid X+Y] + E[Y \mid X+Y] = 1.5$

4. If $X_1, X_2, \dots, X_n$ are i.i.d. $U[0,1]$ RVs.

   (a) Find the pdf of $Y = \min\{X_1, X_2, \dots, X_n\}$. **Answer:** $Pr[Y \geq y] = (1-y)^n$
   Density function is the derivative which is $n(1-y)^{n-1}$.

(b) Let $Z = \max\{X_1, X_2, \ldots, X_{100}\}$. What is $E[Z]$?
Answer: $E[Z] = 100/101$.
This can be done with symmetry on the lengths of the 101 intervals that are the unit interval is cut up into. Or perhaps it one can see it is making a unit necklace, cutting it once and then cutting it uniformly at random. The pieces including the one "at the end" have length 1/101.
One can get the second from the integral of $Pr[Z \geq z] = 1 - (1-z)^{100}$.

5. I want to take a student poll to find the popularity of EECS 70 (assume each student independently likes it with probability p), and I need to pay each student \$1 to get his/her opinion. Suppose I want to estimate p within 1 percent accuracy with a 95% confidence level, I want to find how much money I need to find my estimate.

   (a) What estimator could you use for p from a set of samples, $X_1, X_2, \ldots, X_n$? (It should have expectation p.)
   Answer: Use $A_n = \frac{1}{n}(X_1 + \ldots + X_n)$.

   (b) What is an upper bound on the variance of your estimator that does not depend on p?
   Answer: $(1-p)p/n$ which is upper bounded by $1/4n$

   (c) How much money would I need to spend if I use the CLT?
   Answer:
   The 95% confidence interval for this random variable is $p \pm 2\sigma$ where $\sigma = \sqrt{\frac{p(1-p)}{n}} \leq \sqrt{\frac{1}{4n}}$.
   We wish $2\sqrt{\frac{1}{4n}} \leq .01$ which gives $n \geq (200)^2/4 = 10000$.

   (d) How about if I use the Chebyshev bound?
   Answer: For Chebyshev we get $n \geq (450)^2/4$.

6. A relatively rare disease afflicts 1 in 100 people in the population. Screening for the disease has a missed detection rate of 1% (i.e. there is a 1% chance that a person has the disease but the test doesn't catch it), and a false alarm rate of 5% (i.e. there is a 5% chance that the test comes out positive for the disease even though the person does not have it), then if a test comes out positive, what is the probability that the person has the disease? (The answer can be an expression with numbers, no need to simplify into a single number.)
Answer: Let A indicate disease and B indicate test.
$Pr[A \mid B] = \frac{Pr[B|A]Pr[A]}{Pr[B|A]Pr[A] + Pr[B|\bar{A}]Pr[\bar{A}]} = \frac{.99 * .01}{.99 * .01 + .05 * .99}$.

7. Let X, Y be a pair of random variables. The value of c that minimizes the variance of $X - cY$ is what? (You may refer to any of $E[X], E[Y], Cov(X,Y), Var(X)$, or $Var(Y)$ in your solution.)
Answer: $cov(X,Y)/var(Y)$.

8. Let X, Y, Z be i.i.d. $U[-1,1]$ RVs.

   (a) What is $E[X]$?
   Answer: $E[X] = 0$.

   (b) Find $E[(X+Y+Z)^2 \mid X = x]$.
   Answer: $x^2 + 2/3$
   $E[(X+Y+Z)^2 \mid X] = E[X^2 \mid X] + E[2X(Y+Z) \mid X] + E[(Y+Z)^2 \mid X] = X^2 + 2XE[Y+Z \mid X] + E[Y^2] + 2E[Y]E[Z] + E[Z^2]$
   Using $E[Y+Z], E[Y], E[X] = 0$, and $E[Z^2] = E[Y^2] = 1/3$
   This is then $x^2 + 2/3$

9. The local Safeway has an essentially limitless number of cereal boxes, with each cereal box containing a tiny Marvel Comic superhero figure in it. You win a prize if you can collect 20 distinct superhero figures. Assume that there are a total of 20 distinct superheroes in the collection, with each box equally likely to contain any superhero.

   (a) What is the expected number of cereal boxes you need to buy to win the prize?
   Answer: $\sum_{i=0}^{19} \frac{20}{20-i}$.
   To get the $i+1$th figure, one could consider a geometric distibution where one has $p = (20-i)/20$ probability of success, and the expected number of trials to get a success is $1/p$.

   (b) Now suppose that you have only a $20 budget on the cereal boxes, and each cereal box costs $1. What is the expected number of superheroes you will get with your $20?
   Answer: $20 - 20(19/20)^{20}$.
   This is a balls in bins problem which is asking how many non-empty bins are there when one throws 20 balls into 20 bins.

   (c) What is the variance of the number of distinct superheros that you collect with your $20?
   Answer: The variance of the number of empty bins will equal the variance of the number of full bins, as $X+Y = 20$.
   We can take $Y = X_1 + \cdots + X_{20}$ to be the number of empty bins with X_i being an indicator random variable for ith bin being empty.
   $\mu = E[Y] = 20(19/20)^{20}$ and
   $E[Y^2] = \sum_{ij} E[X_i X_j] = \sum_i E[X_i^2] + \sum_{i \neq j} E[X_i X_j] = 20(19/20)^{20} + 20(19)(18/20)^{20}$.
   The answer is $E[Y^2] - E[Y]^2$, $E[Y] = (20)(19/20)^{20}$.

10. X and Y are continuous random variables and are uniformly distributed with pdf $f(x,y) = c$ over their region of support. Their region of support is the following: $\{1 < X < 2, 1 < Y < 4\} \cup \{2 < X < 3, 2 < Y < 3\}$.

    (a) Find c.
    Answer: $1/4$. The total area of the support is 4.

    (b) Find the marginal distributions of X and Y.
    Answer: $f_X(x) = 3/4$ for $x \in [1,2]$ and $f_X(x) = 1/4$ for $x \in [2,3]$
    $f_Y(y) = 1/4$ for $y \in [1,2]$ $y \in [3,4]$ and $f_Y(y) = 1/2$ for $y \in [2,3]$.

    (c) Find the MMSE (minimum least squares error) estimate of Y given X. (This should not take long, if you don't see it, move on.)
    Answer: $E[Y \mid X] = 2.5$
    For every value of X, the expected value is 2.5.

11. There are N passengers boarding a full flight. They have assigned seats but they have all lost their boarding passes, so they choose to sit in random seats (I know, in real life, they will probably get the aisle or window seats at the front of the plane, but we wanted to keep things simple for you).

    (a) What is the expected number of passengers who sit in their assigned seats?
    Answer: 1.

    (b) What is the probability that i passengers sit in their assigned seats?
    Answer: $\sum_{j=0}^{n-i} (-1)^j \binom{n}{i+j} \frac{(n-(i+j))!}{n!}$
    Inclusion-Exclusion.

12. The lifespans of good batteries are exponentially distributed with mean 2 days. Those of used batteries are exponentially distributed with mean 1 day. There two batches of batteries, one batch has all new batteries, and the other batch has all used batteries, but you don't know which is which. You randomly select one of them and test one battery from the batch, and find that it lasts for 0.75 day.

(a) Let p be the probability that you picked the batch of new batteries. What is p?
Answer:
$Pr[A \mid X = .75] = \frac{Pr[A \cap X = .75]}{Pr[X = .75]} = \frac{Pr[X \in [.75, .75+dx] | A] Pr[A]}{Pr[X \in .75, .75+dx]} = \frac{(e^{-.375}/2dx).5}{(e^{-.375}/2dx).5 + (e^{-.75}dx).5} = \frac{1}{1+2e^{-.375}}$.

(b) What is the expected lifespan of another battery in the batch you picked? (Leave your answer in terms of p, the answer to part (a).)
Answer: $E[X] = 2 * p + 1(1-p) = p + 1$.
$E[X] = E[X \mid batch1] Pr[batch1] + E[X \mid batch2] Pr[batch2]$

13. Let $X \sim expo(\lambda)$, and let $\lceil X \rceil$ denote the ceiling of X - that is, the smallest integer greater than or equal to X. Find the distribution of $\lceil X \rceil$ and identify this distribution as one we have learned in class, with appropriate parameters.
Answer: For $k = 1, 2, ...,$ we have

$$\begin{aligned} P(\lceil X \rceil = k) &= P(k-1 < X \leq k) \\ &= P(X > k-1) - P(X > k) \\ &= e^{-\lambda(k-1)} - e^{-\lambda k} \\ &= \left(e^{-\lambda}\right)^{k-1} (1 - e^{-\lambda}) \end{aligned}$$

Hence $\lceil X \rceil$ is geometric with parameter $1 - e^{-\lambda}$.

14. Let $X = N(0,1)$ and $Y = N(1,1)$ be independent Gaussian random variables. You get an observation $z = 0.6$ that is equally likely to be a realization of either X or Y. You want to decide whether z came from X or Y by evaluating which decision leads to a larger probability of being right.

(a) If you decide z came from X, what is the probabilty that you are right? (No need to evaluate numerically, leave it as a function of the pdf of a standard Normal random variable).
Answer:

$$\begin{aligned} Pr[z \text{ from } N(0,1) \mid z = 0.6] &= \frac{0.5 \times f_X(0.6)}{0.5 \times f_Y(0.6) + 0.5 \times f_X(0.6)} \\ &= \frac{e^{-0.6^2/2}}{e^{-0.4^2/2} + e^{-0.6^2/2}} \end{aligned}$$

(b) Should you decide z is from X or from Y to get a larger probability of being right?
Answer: Y. $e^{-0.4^2/2}$ is bigger than $e^{-0.6^2/2}$.

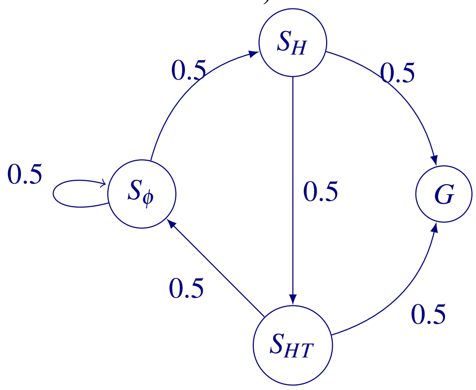
15. A hard-working GSI is holding her office hours (OH) for EECS 70 students. A random number of students enter and leave her office during her OH. Let us break up time into 1-minute segments, and assume that there is either 0 or 1 student in each time segment, and that this discrete arrival/departure process is well modeled by a 2-state Markov Chain. For each time segment, the transition probabilities are 0.8 for going from 0 students to 1 student in the OH, and 0.4 for going from 1 student to 0 students in the OH.

(a) If the GSI starts off her office hours at $t = 0$ with 1 student, what is the probability that she has 0 student at time $t = 2$?
Answer: $0.4 \times 0.2 + 0.6 \times 0.4 = 0.32$.

(b) What does the probability go to, as t gets large, that there is 1 student?
Answer: $0.8p_0 + 0.6p_1 = p_1$ $p_0 = 1 - p_1$
This gives as $0.8(1-p_1) + 0.6p_1 = p_1$ which gives $p_1 = 0.8/1.2 = 2/3$.

16. You flip a fair coin repeatedly until you get 2 Heads in a row or 2 Heads in 3 tosses. We wish to find the expected number of tosses you need before you stop.

    (a) Draw a four state Markov Chain corresponding to this process, where the goal state is terminal (i.e., only has transitions to itself.)



    Answer:

    (b) What is the expected number of tosses you need to stop? (No need to solve the problem numerically, just set up the equations needed to solve it.)

    Answer: $E[X] = 1 + 0.5E[S_H] + 0.5E[X]$, $E[S_H] = 1 + 0.5E[S_{HT}]$, $E[S_{HT}] = 1 + .5E[X]$

7. Probability: Basketball. 15 points.

Alice (A) and Bob (B) play a one-on-one pickup game of basketball. Each made basket counts as 1 point. A beats B by a score of 51-49. We want to find the probability that A leads from start to finish (i.e. the game is never tied at any time other than at the start of the game) but we will lead up to this with some helpful hint questions that should help you find the answer.

Let us first define the following useful events:

T_A : Game had at least one tie and A got the first point;

T_B : Game had at least one tie and B got the first point;

N: Game had no tie..

1. How are $P(T_A), P(T_B)$, and $P(N)$ related?

   Answer: $P(N) + P(T_A) + P(T_B) = 1$

2. Given the final score what is the probability that B got the first point?

   Answer: 49/100.

3. How is $P(T_B)$ related to the probability that B got the first point?

   Answer: $Pr[T_B] = P(B)$ since Alice won.

4. How are $P(T_A)$ and $P(T_B)$ related? (Hint: use symmetry)

   Answer: If you take the sequence to the first tie after the first coin is tossed, the B's and A's are switched and so have the same probability.

   $P(T_A) = P(T_B)$

5. Using the above, What is the probability that A leads from start to finish (i.e. the game is never tied except at the start of the game)?

   Answer: $Pr(N) = 1 - 2 * \frac{49}{100} = 2/100 = 1/50$