CS 70 Fall 2016 — Discrete Mathematics and Probability Theory — Seshia and Walrand — Midterm 1 Solutions

PRINT Your Name: **Answer:** Oski Bear

SIGN Your Name: ____

PRINT Your Student ID: ____

CIRCLE your exam room:
Dwinelle 155   GPB 100   GPB 103   Soda 320   Soda 310   Cory 277   Cory 400   Other

Name of the person sitting to your left: ____

Name of the person sitting to your right: ____

- After the exam starts, please *write your student ID (or name) on every odd page* (we will remove the staple when scanning your exam).
- We will not grade anything outside of the space provided for a problem unless we are clearly told in the space provided for the question to look elsewhere.
- On questions 1-2: You need only give the answer in the format requested (e.g., true/false, an expression, a statement.) We note that an expression may simply be a number or an expression with a relevant variable in it. **For short answer questions, correct clearly identified answers will receive full credit with no justification. Incorrect answers may receive partial credit.**
- On question 3-8, do give arguments, proofs or clear descriptions as requested.
- You may consult one sheet of notes. Apart from that, you may not look at books, notes, etc. Calculators, phones, and computers are not permitted.
- There are **14** single sided pages on the exam. Notify a proctor immediately if a page is missing.
- **You may, without proof, use theorems and lemmas that were proven in the notes and/or in lecture.**
- **You have 120 minutes: there are 8 questions on this exam worth a total of 125 points.**

Do not turn this page until your instructor tells you to do so.

1. **TRUE or FALSE?: total 24 points, each part 3 points**
For each of the questions below, answer TRUE or FALSE.
Clearly indicate your correctly formatted answer: this is what is to be graded.No need to justify!

Answer: Note that the answers provide explanations for your understanding, even though no such justification was required

1. $\forall x \exists y [P(x) \lor Q(y)]$ is equivalent to $[\forall x P(x)] \lor [\exists y Q(y)]$.
Answer: TRUE. $P(x)$ is independent of y, and $Q(y)$ is independent of x. So the quantifiers can be moved inside.

2. If P and Q are propositions, then $(P \lor Q) \Rightarrow (\neg Q)$ is always TRUE.
Answer: FALSE. Simplifies to $\neg Q$ which is FALSE if Q is TRUE.

3. For the Stable Marriage Problem: A female-optimal pairing is male-pessimal.
Answer: TRUE. The notes have a proof showing that a male-optimal pairing is female-pessimal. Switch the two roles.

4. In the Stable Marriage Algorithm (with men proposing), if W is last on every man's preference list and M is not last on any woman's preference list, M cannot end up paired with W.
Answer: FALSE. Counterexample: Consider a 3x3 case:

A	1	2	3
B	2	1	3
C	1	2	3
1	A	C	B
2	B	C	A
3	A	C	B

5. The following statement is a proposition:
"There is a unique integer solution to the equation $x^2 = 4$."
Answer: TRUE. Its value happens to be false (x can be ± 2). Note that x is not an unbound variable here.

6. There exists a graph with 9 vertices, each of degree 3.
Answer: FALSE. The sum of the degrees would be 27 which cannot be $2|E|$.

7. Consider an undirected graph G. If there is a (simple) path in G from vertex x to vertex y through vertex z, and there is a (simple) path in G from y to x through z, then there is a cycle in G containing x, y, and z.
Answer: FALSE. Consider a chain of edges, one end point being x, one being y and z in the middle.

8. If $x \equiv 5 \pmod 9$ and $y \equiv 4 \pmod 9$ then $x + y$ is divisible by 9.
Answer: TRUE. $x = 9k + 5$, $y = 9l + 4$, $x + y = 9(k + l + 1)$.

2. Short Answers: 5x3=15 points Clearly indicate your correctly formatted answer: this is what is to be graded.No need to justify!

1. Write the contrapositive of the following statement: If $x^2 - 3x + 2 = 0$, then $x = 1$ or $x = 2$

   Answer: If $x \neq 1$ and $x \neq 2$, then $x^2 - 3x + 2 \neq 0$.

2. A connected planar simple graph has 5 more edges than it has vertices. How many faces does it have?

   Answer: 7. Use Euler's formula.

3. An n-dimensional hypercube has 2^n vertices. How long can the shortest (simple) path between any two vertices in the hypercube be? (The length of a path is the number of edges in it.)

   Answer: n. Each edge corresponds to flipping a bit. Need to flip at most n bits to get from one bit string to another.

4. TRUE or FALSE: Suppose you are given two trees $T_1 = (V_1, E_1)$ and $T_2 = (V_2, E_2)$ that share no vertices or edges. If you add an edge e connecting some vertex in V_1 to some vertex in V_2, then the resulting graph $(V_1 \cup V_2, E_1 \cup E_2 \cup \{e\})$ is also a tree.

   Answer: TRUE.

5. TRUE or FALSE: In stable marriage, if Man M is at the top of Woman W's ranking but the bottom of every other woman's ranking, then every stable matching must pair M with W.

   Answer: FALSE.

A	1	2
B	2	1
1	A	B
2	B	A

3. Short Proofs: 4+4+4+4+4=20 points

1. Prove that $5\sqrt{2}$ is irrational.

   Answer: Proof by contradiction. Suppose not. Then $5\sqrt{2} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$. Thus, $\sqrt{2} = \frac{a}{5b}$, contradicting the fact that $\sqrt{2}$ is irrational.

2. A Pythagorean triple (a, b, c) has three natural numbers a, b, c such that $a^2 + b^2 = c^2$. Prove that at least one of a, b, c must be even.

   Answer: Proof by contradiction. Assume that none of a, b, and c are even. We will write $a = 2k+1$, $b = 2l+1$, and $c = 2m+1$ where $k, l, m \in \mathbb{N}$. Rewriting the original formula, we get:

   $$c^2 = a^2 + b^2 = (2k+1)^2 + (2l+1)^2 = 4k^2 + 4k + 1 + 4l^2 + 4l + 1 = 2(2k^2 + 2k + 2l^2 + 2l + 1)$$

   Since natural numbers are closed under addition and multiplication, $(2k^2 + 2k + 2l^2 + 2l + 1) \in \mathbb{N}$. Therefore, c^2 must be an even number. However, we also know that $c = 2m+1$, so

   $$c^2 = (2m+1)^2 = 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1$$

   This means that c^2 must be an odd number. Since c^2 cannot be both even and odd, we have reached a contradiction. Therefore, at least one of a, b, and c is even.

3. Prove or disprove: If all vertices of an undirected graph have degree 4, the graph must be the complete graph on 5 vertices, K_5.

   Answer: The statement is FALSE. Consider the 4-dimensional hypercube. Each vertex has exactly 4 neighbors, but it is not K_5.

4. Prove that for any integer n, if n^3+2n+3 is odd, then n is even.

   Answer: Proof by contraposition. Suppose n is an odd integer. We want to show that n^3+2n+3 is even. Since n is odd, we can express it as $n = 2k+1$ for some integer k. It's easy to see that the product of odd integers are odd, or in this case, we can show arithmetically that $n^3 = (2k+1)^3 = 8k^3+12k^2+6k+1$, which is odd. $2n$ is even. The sum of n^3, an odd number, $2n$ and even number, and 3, another odd number, is even. Thus, we proved that "n^3+2n+3 is odd" is false, which concludes the proof.

5. Recall that an *Eulerian walk* in an undirected graph G is a walk in G that traverses each edge exactly once.

   Consider n undirected graphs G_1, G_2, ... G_n that share no vertices or edges and have exactly two odd-degree vertices each. Prove that it is possible to construct an Eulerian tour visiting all of G_1, G_2, ... G_n using only n additional edges to connect them.

   Answer: If G has an Eulerian walk, it must have exactly two vertices with odd degree which are the start and end points of an Eulerian walk. (This was proved in discussion section, but easy to show.)

   We can prove this using induction.

   Consider the base case to be one graph G_1. We can use the one edge to connect its two odd-degree vertices.

   To convince ourselves, we can consider a second base case, G_1, G_2. So let these vertices (start and end points of the Eulerian walk) be v_{1s} and v_{1t} in G_1, and v_{2s} and v_{2t} in G_2. Connect v_{1t} to v_{2s} and v_{2t} to v_{1s}. That yields an Eulerian tour for the whole graph.

   The inductive step follows, where we link each G_i to the G_{i+1} graph by connecting their odd-degree vertices, and the G_n graph to G_1 in the same manner.

4. Checking Proofs:3+3= 6 points

Each of the proofs below has a fallacy on a single line. Find the fallacy, and explain your answer briefly.

1. Proposition: For any integers x, y, and n, if $x-y$ is divisible by n, then so is $x+y$.

   Proof: If $x-y$ is divisible by n, then we can write $x-y \equiv 0 \pmod{n}$ or $x \equiv y \pmod{n}$.
   Squaring both sides, we get $x^2 \equiv y^2 \pmod{n}$.
   Taking square roots, we get $x \equiv -y \pmod{n}$.
   Rewriting, we get $x+y \equiv 0 \pmod{n}$, or $x+y$ is divisible by n. □
   Answer: Cannot "take square roots" in the way we did.

2. Proposition: Let a be a two digit (decimal) number and b be formed by reversing the digits of a. Then the digits of a^2 are simply those of b^2 reversed.
   (For example, if $a = 10$, $b = 01$, then $a^2 = 100, b^2 = 001$. Similarly, if $a = 12$,$b = 21$, we have $a^2 = 144, b^2 = 441$.)

   Proof: Let $a = 10x+y$ where x, y are decimal digits. Then $b = 10y+x$.
   This gives us:
   $$a^2 = 100x^2 + 20xy + y^2 = 100x^2 + 10(2xy) + y^2$$
   $$b^2 = 100y^2 + 20yx + x^2 = 100y^2 + 10(2yx) + x^2$$
   Thus, the digits of a^2 are x^2, $2xy$, and y^2 and similarly the digits of b^2 are y^2, $2yx$, and x^2, exactly reverse.
   This yields the desired result. □
   Answer: Cannot conclude that digits of a^2 are x^2, $2xy$, and y^2 (and similarly of b^2) when $(x^2 \vee 2xy \vee y^2) > 9$.

5. Proofs about XOR: 3+7= 10 points

Recall from Homework 1 the XOR operator, written $\oplus$: $P \oplus Q$ is TRUE if and only if exactly one of P and Q is TRUE and the other is FALSE.

1. Show that $\oplus$ is associative: given three propositions P_1, P_2, P_3, that $P_1 \oplus (P_2 \oplus P_3) \equiv (P_1 \oplus P_2) \oplus P3$.

   Answer: One way to solve is using a truth table.

2. Now, given n propositions P_1, P_2, $\ldots$, P_n, $n \geq 2$, we can construct the XOR of all of them: $P_1 \oplus P_2 \oplus P_3 \oplus \ldots \oplus P_n$. (Since $\oplus$ is associative, it does not matter how we put parentheses around them, so we omit this.) Call this Q_n; that is, $Q_n = P_1 \oplus P_2 \oplus P_3 \oplus \ldots \oplus P_n$.

   A *satisfying assignment* to Q_n is an assignment of TRUE/FALSE to the propositions $P_1, P_2, \ldots, P_n$ such that Q_n is TRUE. A *falsifying assignment* to Q_n is a TRUE/FALSE assignment to the P_is such that Q_n is FALSE.

   Prove that for all n, Q_n has exactly 2^{n-1} satisfying assignments.

   Answer: Base case: $n = 2$: Exactly 2 assignments.

   Induction hypothesis: Q_k has exactly 2^{k-1} satisfying assignments.

   Induction step: Assuming the induction hypothesis, we need to show that Q_{k+1} has exactly 2^k satisfying assignments.

   Notice that $Q_{k+1} = Q_k \oplus P_{k+1}$. Consider an arbitrary satisfying assignment to Q_k, say a_k. To this, add $P_{k+1} = FALSE$. Then $(a_k, FALSE)$ is a satisfying assignment to Q_{k+1}. But $(a_k, TRUE)$ is a falsifying assignment. Similarly, for every falsifying assignment b_k to Q_k, adding $P_{k+1} = TRUE$ makes it a satisfying assigment, whereas $P_{k+1} + FALSE$ makes it a falsifying assignment.

   Thus, the total number of satisfying assignments are $2^{k-1} + 2^{k-1} = 2^k$.

6. Perfect Matching in Graphs: 8+4+8=20 points

For an undirected (simple) graph with n vertices, where n is even, a *perfect matching* is a set of $n/2$ edges such that every vertex of the graph is incident to exactly one of the edges in the set.

1. Prove or disprove: Every tree has at most one perfect matching.

   Answer: True.

   Method 1: Let M, M' be perfect matchings in the tree $T = (V, E)$ and consider the graph on V with edge set $M \cup M'$. Since M and M' both cover all the vertices, every connected component of this new graph is either a single edge (common to both M and M') or a cycle. Since T is a tree, there can be no cycle, so we conclude that $M = M'$.

   Method 2: Proof by induction. Strengthen the inductive hypothesis to say that every forest with at most k vertices has at most one perfect matching. We have for $n = 1$, no perfect matchings exist and for $n = 2$, exactly one perfect matching exists (if the nodes are connected) or no perfect matchings (if they are disconnected). Consider any forest on $k+1$ nodes. There exists some leaf node l, which in any perfect matching must be matched with its parent node (because that is the only edge incident to l). Delete l and its parent and all incident edges to those nodes from the forest. We are left with another forest (we cannot create any cycles by deleting edges and nodes), in which every tree has $\leq k-1$ nodes. We know by the inductive hypothesis that each of these trees has at most one perfect matching, so the original forest has at most one perfect matching (the unique perfect matchings of each tree in the forest and the edge connecting l to its parent). **Note**: Can you figure out how to make the induction work if you do not strengthen the inductive hypothesis? □

2. Prove or disprove: If a graph has a perfect matching, it is 2-colorable. (That is, each vertex can be assigned one of two colors so that no two adjacent vertices have the same color.)

   Answer: False. Consider K_4. □

3. Prove that if G has the following property P:

$$G \text{ is a simple graph with } 2n\ (n \geq 2) \text{ vertices such that every vertex has degree} \geq n$$

then G has a perfect matching.

(Hint: Prove that all graphs satisfying P have a Hamiltonian cycle; we suggest a proof by contradiction for this. Recall that a Hamiltonian cycle is one that visits each vertex exactly once.)

Answer: We will show that all graphs satisfying P have a Hamiltonian cycle. Once we have shown this, we can take every second edge of the Hamiltonian cycle as our perfect matching.

Suppose that the above statement is false, i.e. there is a graph satisfying P which does not have a Hamiltonian cycle. Let G' be the largest such counterexample, in the sense that G' has the greatest number of edges out of any counterexample.

Claim: G' has a Hamiltonian path.

Proof: If $G' = (V,E)$ does not have a Hamiltonian path, consider the longest possible path $v_0, \dots, v_k$. There must be some vertex u that the longest path does not visit. Consider the graph $G'' = (V, E \cup \{v_k, u\})$ formed by adding the edge $\{v_k, u\}$ to G'. Adding this edge cannot create a Hamiltonian cycle (otherwise, its removal would imply that G' had a Hamiltonian path). We see that G'' is a graph satisfying P with more edges than G', which contradicts our choice of G' as the largest counterexample. This establishes the claim: G' must have a Hamiltonian path.

Claim: G' has a Hamiltonian cycle.

Proof: Let $v_1, \dots, v_{2n}$ be the Hamiltonian path of G'. If $\{v_1, v_{2n}\}$ is an edge, then we are done: G' has a Hamiltonian cycle. Otherwise, v_1 has n neighbors in $v_2, \dots, v_{2n-1}$, and so does v_{2n}. This is only possible if there exists an index i such that v_{2n} is adjacent to v_i and v_1 is adjacent to v_{i+1}. Then, $v_1, v_2, \dots, v_i, v_{2n}, v_{2n-1}, \dots, v_{i+1}$ is our Hamiltonian cycle.

However, G' was supposed to be a counterexample! Contradiction. □

Remark: The statement "if G is a simple graph with $2n$, $n \geq 2$, where every vertex has degree at least n, then G has a Hamiltonian cycle" is a theorem first proven by Dirac. There are a few other ways to prove this statement. One way is to induct on the length of the longest path of the graph, showing that every path can be extended into a Hamiltonian cycle. (The details are tricky.) Another method is to consider the longest path, show that the path can be made into a cycle, and then to prove that this cycle must be Hamiltonian. (The argument goes: pick a vertex lying outside of this cycle, show that the vertex is connected to the cycle, but then the path which starts at this vertex and follows the cycle is a longer path than the one we initially considered.) Our choice of G' as the largest counterexample eliminated some of the complications of the proof, effectively reducing the proof to the implication "G has a Hamiltonian path and G satisfies P" $\Rightarrow$ "G has a Hamiltonian cycle".

7. Stable Marriage Problem: 3+7=10 points

Consider the following stable marriage instance.

Man	Women			
A	2	4	1	3
B	3	1	4	2
C	1	4	2	3
D	3	4	2	1

Woman	Men			
1	A	C	B	D
2	B	C	D	A
3	B	A	C	D
4	B	A	D	C

1. List all the rogue couples for the following pairing: (A,1), (B,2), (C,3), (D,4)

   Answer: (B,3),(A,4),(B,4)

2. For each woman, find her optimal man and her pessimal man. Show all your work and justify your answer.

   Answer: Run the algorithm with women proposing (the algorithm is now female-optimal, so it matches each woman to her optimal man):

	Day 1	Day 2	Day 3	Day 4
A	①	1 ④	④	④
B	2 ③ 4	③	③	③
C		②	① 2	①
D				②

Now run the algorithm with men proposing (the algorithm is male-optimal and female-pessimal, so it matches each woman to her pessimal man):

	Day 1	Day 2
1	Ⓒ	Ⓒ
2	Ⓐ	Ⓐ
3	Ⓑ D	Ⓑ
4		Ⓓ

Final Answer:

Woman	Optimal Man	Pessimal Man
1	C	C
2	D	A
3	B	B
4	A	D

8. Boolean Division: 10+10=20 points

Given predicates $F(x)$ and $D(x)$, we say that $D(x)$ is a *Boolean divisor* of $F(x)$ if there exist predicates $Q(x)$ and $R(x)$ such that $\forall x,\ F(x) = \{[D(x) \wedge Q(x)] \vee R(x)\}$, where $\exists x, \{D(x) \wedge Q(x) \neq \text{FALSE}\}$.
(In other words, a Boolean divisor is like integer division, where multiplication is replaced by AND, and addition by OR. Also note that we use "=" to mean propositional equivalence.)

A predicate $D(x)$ of $F(x)$ is said to be a *factor* of $F(x)$ if there exists a predicate $Q(x)$ such that $\forall x,\ F(x) = [D(x) \wedge Q(x)]$.

[Hint for both parts below: try using identities that simplify propositional forms.]

1. Prove that for any two predicates $F(x)$ and $D(x)$, $D(x)$ is a factor of $F(x)$ if and only if $\forall x, \{F(x) \wedge (\neg D(x)) = \text{FALSE}\}$.

   Answer: Only if part is easy: $F(x) \wedge (\neg D(x))$ simplifies to FALSE for all x.

   If part: Write $F(x) = F(x) \wedge [D(x) \vee \neg D(x)] = [F(x) \wedge D(x)] \vee [F(x) \wedge \neg D(x)]$. The second term on the RHS simplifies to FALSE, and we can take $Q(x) = F(x)$ to yield the desired result.

2. Prove that for any two predicates $F(x)$ and $D(x)$, $D(x)$ is a Boolean divisor of $F(x)$ if and only if $\exists x, \{F(x) \wedge D(x) \neq \text{FALSE}\}$.

   Answer: (only if part): $\forall x, F(x) = [D(x) \wedge Q(x)] \vee R(x)$. Thus, for all x, $F(x) \wedge D(x) = [D(x) \wedge Q(x)] \vee [D(x) \wedge R(x)]$. Since $\exists x,\ D(x) \wedge Q(x) \neq \text{FALSE}$, there must exists x such that $F(x) \wedge D(x) \neq \text{FALSE}$.

   (if part): Rewrite $F(x)$ as $[F(x) \wedge D(x)] + [F(x) \wedge \neg D(x)]$. Take $Q(x) = F(x)$, which is OK, since $\exists x, F(x) \wedge D(x) \neq \text{FALSE}$. Also take $R(x) = [F(x) \wedge \neg D(x)]$. This proves the result.

(Scratch space)