CS 70 Discrete Mathematics and Probability Theory
Fall 2011 Rao Midterm 2 Solutions

1 True/False. [24 pts]

Circle one of the provided answers please!
No negative points will be assigned for incorrect answers.

(a) TRUE or FALSE: Given independent events A, B where A and B have nonzero probability, then $A \cap B$ is nonempty.

(b) TRUE or FALSE: If A, B, and C are mutually independent, then $Pr[A|B,C] = Pr[A]$.

(c) TRUE or FALSE: If $Pr[A|B] = 2Pr[A]$, then $Pr[B] > Pr[A]$.

(d) TRUE or FALSE: It is necessarily true that the variance of a random variable X is $\leq (E(X))^2$.

(e) TRUE or FALSE: It is necessarily true that the variance of a random variable X is $\leq E(X^2)$.

(f) TRUE or FALSE: For disjoint events A and B, the $Pr[A \cap B] = Pr[A] \times Pr[B]$.

(g) TRUE or FALSE: For independent events, $Pr[A \cup B] = Pr[A] + Pr[B]$.

(h) TRUE or FALSE: For a Poisson random variable X with parameter λ, the $Pr[X = i+1] \leq Pr[X = i]$ for all $i \geq \lambda$.

(i) TRUE or FALSE: For a Poisson random variable X with parameter $\lambda = 1$, then Chebyshev's inequality ensures that the $Pr[X \geq 11] \leq \frac{1}{100}$.

(j) TRUE or FALSE: For a binomially distributed variable X with parameter $p = \frac{1}{2}$ and $n = 100$, Chebyshev's inequality ensures that the $Pr[X \geq 75] \leq \frac{1}{10}$.

(k) TRUE or FALSE: Given two random variables, X with Poisson distribution and Y with a geometric distribution, both with mean μ, we can conclude that $E[X+Y] > E[2X]$.

(l) TRUE or FALSE: The maximum variance binomial distribution with parameter n has parameter $p = 1$.

(m) TRUE or FALSE: Given a random variable $S = X_1 + \ldots + X_n$ where the X_i's are chosen independently from the same distribution, and any α, $Pr[|S - E[S]| \geq \alpha]$ goes to 0 as n goes to infinity.

2 Short answer. [43 pts]

For parts a to b, consider a deck with just the four aces (red: hearts, diamonds; black: spades, clubs). Melissa shuffles the deck and draws the top two cards.

(a) [4 pts] Given that Melissa has the ace of hearts, what is the probability that Melissa has both red cards?

(b) [4 pts] Given that Melissa has at least one red card, what is the probability that she has both red cards?

(c) [4 pts] Suppose that A and B are independent, C is disjoint from both A and B and $P[A] = P[B] = P[C] = 1/4$. Compute $P[A \cup B \cup C]$.

For parts d to h, we consider two events A and B such that P(A) = 0.3 and P(B) = 0.4. Compute P(A|B) in each of the following cases:

(d) [3 pts] A and B are independent

(e) [3 pts] A and B are disjoint

(f) [3 pts] $A \implies B$

(g) $P[A \cap B] = 0.1$

(h) [3 pts] $P(A \cup B) = 0.5$

(i) [4 pts] The number of accidents (per month) at a certain factory has a Poisson distribution. If the probability that there is at least one accident is 1/2, what is the probability that there are exactly two accidents?

(j) [4 pts] A pair of dice is rolled until either a 4 is rolled (the numbers on the two dice add up to 4) or a 7 is rolled. What is the expected number of rolls needed?

(k) [4 pts] There is a test to determine whether one has boneitis, but the test is not always accurate. For those who do have boneitis, the test has an 4 in 5 chance of coming out positive. For those who don't have boneitis, the test has a 1 in 9 chance of coming out positive. Overall, about 10% of people have boneitis.

Suppose the test comes out positive for That Guy. What is the probability That Guy has boneitis?

(l) [4 pts] A hand of 13 cards are chosen (without replacement) at random from a standard deck of 52 poker cards. What is the expected number of four-of-a-kinds that we see from these 13 cards? (No need to evaluate the expression to get a number.)

(Four-of-a-kind is four cards of the same rank. For example, the hand
"A♠A♠A♢A♣K♠K♡K♢K♣2♠2♠2♢2♣6♢"
contains three four-of-a-kinds, namely the aces, the kings and the twos.)

3 3-SAT. [15 pts]

A 3-conjuctive normal form (CNF) formula is a boolean formula consisting of the "and" of a sequence of clauses where each clause consists of the "or" of three literals. For example, $\phi = (x_1 \vee x_2 \vee \overline{x_5}) \wedge (x_5 \vee \overline{x_2} \vee \overline{x_1})$. (No variable can appear twice in a single clause.)

One wishes to find an assignment to the variables to maximize the number of true clauses. The literals work in the natural manner: $x_1 = T$ if and only if $\overline{x_1} = F$. In the example above, the assignment, $x_1 = T, x_2 = T$ and $x_5 = F$ satisfies one clause in ϕ, where $x_1 = T, x_2 = F, x_5 = F$ satisfies two clauses in ϕ.

(a) For a particular formula with n clauses, consider choosing a random assignment to the variables, i.e., $x_i = T$ or $x_i = F$ with equal probability. What is the expected number of satisfied clauses?

(b) Let U be a random variable corresponding to the number of unsatisfied clauses. What is $E(U)$?

(c) Upper bound the probability that U is larger than $(1+\varepsilon)E(U)$ for $\varepsilon \geq 0$ as a function of ε. (You should give a nontrivial bound here.)

(d) Consider repeating this experiment until one finds an assignment that leaves at most $(1+\varepsilon)E(U)$ unsatisfied clauses. Give an upper bound on the expected number of repetitions.

4 The evolution of a social network. [18 pts]

(We give a simplified analysis of the connectivity of a social network.)

Say one person in a class of n people knows a secret, perhaps where the midterm is. Occasionally a randomly chosen person A **who doesn't know the secret** calls a randomly chosen person B ($B \neq A$) and learns the secret if B knows it.

Let X_2 be a random variable that represents the number of calls (no two calls are simultaneous) until two people know the secret.

(a) What is the distribution of X_2?

(b) What is $E[X_2]$?

(c) Let X_i be the number of calls needed to go from $i-1$ people knowing the secret to i people. What is $E[X_i]$?

(d) What is the expected time for everyone to know the secret?

(e) Bound your expression to within a constant factor for large n. Your expression should not have a summation. (You may use $\Theta(\cdot)$ notation, recall that $2n^2 - 5n + 2 = \Theta(n^2)$.)