

## Next up: how big is infinity.

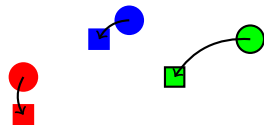
- ▶ Countable
- ▶ Countably infinite.
- ▶ Enumeration

How big are the reals or the integers?

Infinite!

Is one bigger or smaller?

## Same size?



Same number?

Make a function  $f : \text{Circles} \rightarrow \text{Squares}$ .

$f(\text{red circle}) = \text{red square}$

$f(\text{blue circle}) = \text{blue square}$

$f(\text{circle with black border}) = \text{square with black border}$

One to one. Each circle mapped to different square.

One to One: For all  $x, y \in D$ ,  $x \neq y \implies f(x) \neq f(y)$ .

Onto. Each square mapped to from some circle .

Onto: For all  $s \in R$ ,  $\exists c \in D, s = f(c)$ .

**Isomorphism principle:** If there is  $f : D \rightarrow R$  that is one to one and onto, then,  $|D| = |R|$ .

# Isomorphism principle.

Given a function,  $f : D \rightarrow R$ .

## **One to One:**

For all  $\forall x, y \in D, x \neq y \implies f(x) \neq f(y)$ .

or

$\forall x, y \in D, f(x) = f(y) \implies x = y$ .

**Onto:** For all  $y \in R, \exists x \in D, y = f(x)$ .

$f(\cdot)$  is a **bijection** if it is one to one and onto.

## **Isomorphism principle:**

If there is a bijection  $f : D \rightarrow R$  then  $|D| = |R|$ .

# Countable.

How to count?

0, 1, 2, 3, ...

The Counting numbers.

The natural numbers!  $N$

Definition:  $S$  is **countable** if there is a bijection between  $S$  and some subset of  $N$ .

If the subset of  $N$  is finite,  $S$  has finite **cardinality**.

If the subset of  $N$  is infinite,  $S$  is **countably infinite**.

## Where's 0?

Which is bigger?

The positive integers,  $\mathbb{Z}^+$ , or the natural numbers,  $\mathbb{N}$ .

Natural numbers.  $0, 1, 2, 3, \dots$

Positive integers.  $1, 2, 3, \dots$

Where's 0?

More natural numbers!

Consider  $f(z) = z - 1$ .

For any two  $z_1 \neq z_2 \implies z_1 - 1 \neq z_2 - 1 \implies f(z_1) \neq f(z_2)$ .

One to one!

For any natural number  $n$ , for  $z = n + 1$ ,  $f(z) = (n + 1) - 1 = n$ .

Onto for  $\mathbb{N}$

Bijection!  $\implies |\mathbb{Z}^+| = |\mathbb{N}|$ .

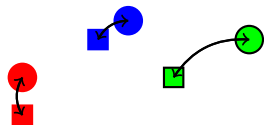
But.. but Where's zero? "Comes from 1."

# A bijection is a bijection.

Notice that there is a bijection between  $N$  and  $Z^+$  as well.

$$f(n) = n + 1. \quad 0 \rightarrow 1, 1 \rightarrow 2, \dots$$

Bijection from  $A$  to  $B \implies$  a bijection from  $B$  to  $A$ .



Inverse function!

Can prove equivalence either way.

Bijection to or from natural numbers implies countably infinite.

## More large sets.

$E$  - Even natural numbers?

$f: \mathbb{N} \rightarrow E$ .

$f(n) \rightarrow 2n$ .

Onto:  $\forall e \in E, f(e/2) = e$ .  $e/2$  is natural since  $e$  is even

One-to-one:  $\forall x, y \in \mathbb{N}, x \neq y \implies 2x \neq 2y. \equiv f(x) \neq f(y)$

Evens are countably infinite.

Evens are same size as all natural numbers.



## All integers?

What about Integers,  $Z$ ?

Define  $f : N \rightarrow Z$ .

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if } n \text{ odd.} \end{cases}$$

One-to-one: For  $x \neq y$

if  $x$  is even and  $y$  is odd,

then  $f(x)$  is nonnegative and  $f(y)$  is negative  $\implies f(x) \neq f(y)$

if  $x$  is even and  $y$  is even,

then  $x/2 \neq y/2 \implies f(x) \neq f(y)$

....

Onto: For any  $z \in Z$ ,

if  $z \geq 0$ ,  $f(2z) = z$  and  $2z \in N$ .

if  $z < 0$ ,  $f(2|z| - 1) = z$  and  $2|z| + 1 \in N$ .

Integers and naturals have same size!

## Listings..

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if } n \text{ odd.} \end{cases}$$

### Another View:

$n$	$f(n)$
0	0
1	-1
2	1
3	-2
4	2
...	...

Notice that: A listing “is” a bijection with a subset of natural numbers.

Function  $\equiv$  “Position in list.”

If finite: bijection with  $\{0, \dots, |S| - 1\}$

If infinite: bijection with  $N$ .

## Enumerability $\equiv$ countability.

Enumerating (listing) a set implies that it is countable.

“Output element of  $S$ ”,

“Output next element of  $S$ ”

...

Any element  $x$  of  $S$  has *specific, finite* position in list.

$Z = \{0, 1, -1, 2, -2, \dots\}$

$Z = \{\{0, 1, 2, \dots\} \text{ and then } \{-1, -2, \dots\}\}$

When do you get to  $-1$ ? at infinity?

Need to be careful.

61A — streams!

## Countably infinite subsets.

Enumerating a set implies countable.

Corollary: Any subset  $T$  of a countable set  $S$  is countable.

Enumerate  $T$  as follows:

Get next element,  $x$ , of  $S$ ,  
output only if  $x \in T$ .

Implications:

$\mathbb{Z}^+$  is countable.

It is infinite since the list goes on.

There is a bijection with the natural numbers.

So it is countably infinite.

All countably infinite sets have the same cardinality.

## Enumeration example.

All binary strings.

$$B = \{0, 1\}^*.$$

$$B = \{\phi, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, \dots\}.$$

$\phi$  is empty string.

For any string, it appears at some position in the list.

If  $n$  bits, it will appear before position  $2^{n+1}$ .

Should be careful here.

$$B = \{\phi, ;, 0, 00, 000, 0000, \dots\}$$

Never get to 1.

## More fractions?

Enumerate the rational numbers in order...

$0, \dots, 1/2, \dots$

Where is  $1/2$  in list?

After  $1/3$ , which is after  $1/4$ , which is after  $1/5$ ...

A thing about fractions:

any two fractions has another fraction between it.

Can't even get to "next" fraction!

Can't list in "order".

## Pairs of natural numbers.

Consider pairs of natural numbers:  $N \times N$

E.g.: (1,2), (100,30), etc.

For finite sets  $S_1$  and  $S_2$ ,

then  $S_1 \times S_2$

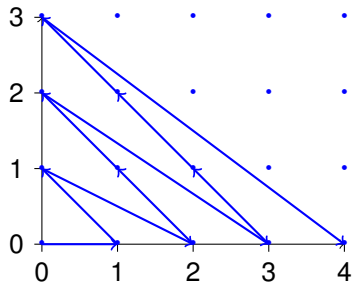
has size  $|S_1| \times |S_2|$ .

So,  $N \times N$  is countably infinite squared ???

# Pairs of natural numbers.

Enumerate in list:

$(0,0), (1,0), (0,1), (2,0), (1,1), (0,2), \dots$



The pair  $(a, b)$ , is in first  $\approx (a + b + 1)(a + b)/2$  elements of list!  
(i.e., “triangle”).

Countably infinite.

Same size as the natural numbers!!



## Rationals?

Positive rational number.

Lowest terms:  $a/b$

$a, b \in \mathbb{N}$

with  $\gcd(a, b) = 1$ .

Infinite subset of  $\mathbb{N} \times \mathbb{N}$ .

Countably infinite!

All rational numbers?

Negative rationals are countable. (Same size as positive rationals.)

Put all rational numbers in a list.

First negative, then nonnegative ??? No!

Repeatedly and alternatively take one from each list.

Interleave Streams in 61A

The rationals are countably infinite.

## Real numbers..

Real numbers are same size as integers?

# The reals.

Are the set of reals countable?

Lets consider the reals  $[0, 1]$ .

Each real has a decimal representation.

.500000000...  $(1/2)$

.785398162...  $\pi/4$

.367879441...  $1/e$

.632120558...  $1 - 1/e$

.345212312... Some real number

# Diagonalization.

If countable, there a listing,  $L$  contains all reals. For example

0: .500000000...

1: .785398162...

2: .367879441...

3: .632120558...

4: .345212312...

⋮

Construct “diagonal” number: .77677...

Diagonal Number: Digit  $i$  is 7 if number  $i$ 's  $i$ th digit is not 7  
and 6 otherwise.

Diagonal number for a list differs from every number in list!

Diagonal number not in list.

Diagonal number is real.

Contradiction!

Subset  $[0, 1]$  is not countable!!

## All reals?

Subset  $[0, 1]$  is not countable!!

What about all reals?

No.

Any subset of a countable set is countable.

If reals are countable then so is  $[0, 1]$ .

# Diagonalization.

1. Assume that a set  $S$  can be enumerated.
2. Consider an arbitrary list of all the elements of  $S$ .
3. Use the diagonal from the list to construct a new element  $t$ .
4. Show that  $t$  is different from all elements in the list  
 $\implies t$  is not in the list.
5. Show that  $t$  is in  $S$ .
6. Contradiction.

## Another diagonalization.

The set of all subsets of  $N$ .

Example subsets of  $N$ :  $\{0\}$ ,  $\{0, \dots, 7\}$ ,  
evens, odds, primes,

Assume is countable.

There is a listing,  $L$ , that contains all subsets of  $N$ .

Define a diagonal set,  $D$ :

If  $i$ th set in  $L$  does not contain  $i$ ,  $i \in D$ .  
otherwise  $i \notin D$ .

$D$  is different from  $i$ th set in  $L$  for every  $i$ .

$\implies D$  is not in the listing.

$D$  is a subset of  $N$ .

$L$  does not contain all subsets of  $N$ .

Contradiction.

**Theorem:** The set of all subsets of  $N$  is not countable.  
(The set of all subsets of  $S$ , is the **powerset** of  $N$ .)

## Diagonalize Natural Number.

Natural numbers have a listing,  $L$ .

Make a diagonal number,  $D$ :  
differ from  $i$ th element of  $L$  in  $i$ th digit.

Differs from all elements of listing.

$D$  is a natural number... **Not.**

Any natural number has a finite number of digits.

“Construction” requires an infinite number of digits.



# The Continuum hypothesis.

There is no set with cardinality between the naturals and the reals.

First of Hilbert's problems!

# Cardinalities of uncountable sets?

Cardinality of  $[0, 1]$  smaller than all the reals?

$f: \mathbb{R}^+ \rightarrow [0, 1]$ .

$$f(x) = \begin{cases} x + \frac{1}{2} & 0 \leq x \leq 1/2 \\ \frac{1}{4x} & x > 1/2 \end{cases}$$

One to one.  $x \neq y$

If both in  $[0, 1/2]$ , a shift  $\implies f(x) \neq f(y)$ .

If neither in  $[0, 1/2]$  a division  $\implies f(x) \neq f(y)$ .

If one is in  $[0, 1/2]$  and one isn't, different ranges  $\implies f(x) \neq f(y)$ .

Bijection!

$[0, 1]$  is same cardinality as nonnegative reals!

## Generalized Continuum hypothesis.

There is no infinite set whose cardinality is between the cardinality of an infinite set and its power set.

The powerset of a set is the set of all subsets.

# Resolution of hypothesis?

Gödel. 1940.

Can't use math!

If math doesn't contain a contradiction.

This statement is a lie.

Is the statement above true?

The barber shaves every person who does not shave themselves.

Who shaves the barber?

Self reference.

Can a program refer to a program?

Can a program refer to itself?

Uh oh....