Today

Random Variables.

Quick Review: Probability. Some Rules.

- **Sample Space:** Set of outcomes, Ω.
- **Probability:** $Pr[\omega]$ for all $\omega \in \Omega$.
  - $0 \leq Pr[\omega] \leq 1$.
  - $\sum_{\omega \in \Omega} Pr[\omega] = 1$.
- **Event:** $A \subseteq \Omega$. $Pr[A] = \sum_{\omega \in A} Pr[\omega]$
  - Inclusion/Exclusion: $Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B]$.
  - Simple Total Probability: $Pr[B] = Pr[A \cap B] + Pr[\overline{A} \cap B]$.
  - Complement: $Pr[\overline{A}] = 1 - Pr[A]$.
  - Union Bound. Total Probability.
- **Conditional Probability:** $Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}$
- Bayes' Rule: $Pr[A_m|B] = p_m q_m / (p_1 q_1 + \cdots + p_M q_M)$.
- Product Rule:
  $Pr[A_1 \cap \cdots \cap A_n] = Pr[A_1] Pr[A_2|A_1] \cdots Pr[A_n|A_1 \cap \cdots \cap A_{n-1}]$.
- Total Probability/Product: $Pr[B] = Pr[B|A] Pr[A] + Pr[B|\overline{A}] Pr[\overline{A}]$.

Random Variables

$\boxed{\text{Random Variables}}$

1. Random Variables.
2. Expectation
3. Distributions.

Questions about outcomes ...

Experiment: roll two dice.
Sample Space: $\{(1,1),(1,2),\ldots,(6,6)\} = \{1,\ldots,6\}^2$
How many pips?

Experiment: flip 100 coins.
Sample Space: $\{HHH\cdots H, THH\cdots H,\ldots,TTT\cdots T\}$
How many heads in 100 coin tosses?

Experiment: choose a random student in cs70.
Sample Space: $\{Adam, Jin, Bing,\ldots,Angeline\}$
What midterm score?

Experiment: hand back assignments to 3 students at random.
Sample Space: $\{123,132,213,231,312,321\}$
How many students get back their own assignment?

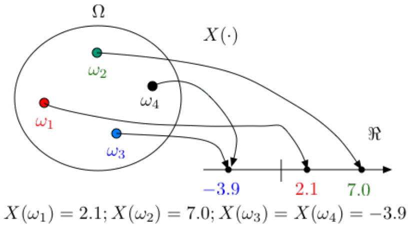
In each scenario, each outcome gives a number.

The number is a (known) function of the outcome.

Random Variables.

A **random variable,** X**,** for an experiment with sample space Ω is a function $X : \Omega \to \Re$.

Thus, $X(\cdot)$ assigns a real number $X(\omega)$ to each $\omega \in \Omega$.



Function $X(\cdot)$ defined on outcomes Ω.

Function $X(\cdot)$ is not random, not a variable!

What varies at random (among experiments)? The outcome!

Note:Random variable induces partition:
$A_y = \{\omega \in \Omega : X(\omega) = y\} = X^{-1}(y)$

Example 1 of Random Variable

Experiment: roll two dice.
Sample Space: $\{(1,1),(1,2),\ldots,(6,6)\} = \{1,\ldots,6\}^2$
Random Variable X: number of pips.
$X(1,1) = 2$
$X(1,2) = 3,$
$\vdots$
$X(6,6) = 12,$
$X(a,b) = a + b, (a,b) \in \Omega.$

Example 2 of Random Variable

Experiment: flip three coins

Sample Space: $\{HHH, THH, HTH, TTH, HHT, THT, HTT, TTT\}$

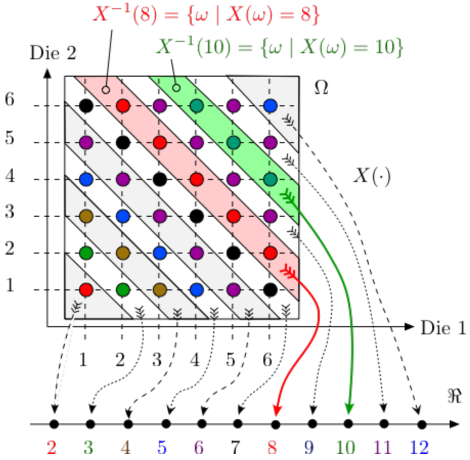
Winnings: if win 1 on heads, lose 1 on tails: X

$X(HHH) = 3 \quad X(THH) = 1 \quad X(HTH) = 1 \quad X(TTH) = -1$

$X(HHT) = 1 \quad X(THT) = -1 \quad X(HTT) = -1 \quad X(TTT) = -3$

Number of pips in two dice.

"What is the likelihood of getting *n* pips?"

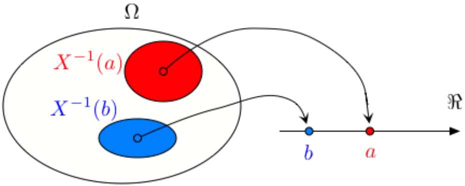


$$Pr[X = 10] = 3/36 = Pr[X^{-1}(10)]; Pr[X = 8] = 5/36 = Pr[X^{-1}(8)].$$

Distribution

The probability of X taking on a value a.

Definition: The **distribution** of a random variable X, is $\{(a, Pr[X = a]) : a \in \mathscr{A}\}$, where $\mathscr{A}$ is the range of X.



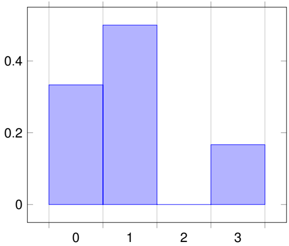
$$Pr[X = a] := Pr[X^{-1}(a)] \text{ where } X^{-1}(a) := \{\omega \mid X(\omega) = a\}.$$

Handing back assignments

Experiment: hand back assignments to 3 students at random.
Sample Space: $\Omega = \{123, 132, 213, 231, 312, 321\}$
How many students get back their own assignment?
Random Variable: values of $X(\omega) : \{3, 1, 1, 0, 0, 1\}$

Distribution:

$$X = \begin{cases} 0, & \text{w.p. } 1/3 \\ 1, & \text{w.p. } 1/2 \\ 3, & \text{w.p. } 1/6 \end{cases}$$

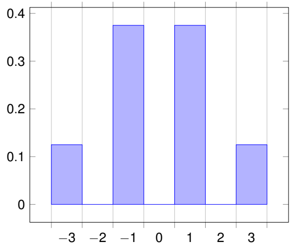


Flip three coins

Experiment: flip three coins
Sample Space: $\{HHH, THH, HTH, TTH, HHT, THT, HTT, TTT\}$
Winnings: if win 1 on heads, lose 1 on tails. X
Random Variable: $\{3, 1, 1, -1, 1, -1, -1, -3\}$

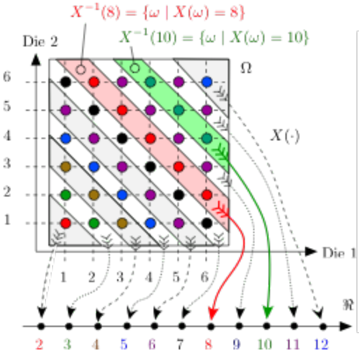
Distribution:

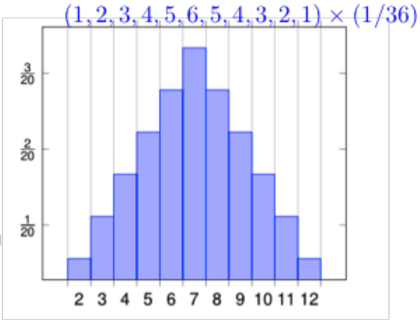
$$X = \begin{cases} -3, & \text{w. p. } 1/8 \\ -1, & \text{w. p. } 3/8 \\ 1, & \text{w. p. } 3/8 \\ 3 & \text{w. p. } 1/8 \end{cases}$$



Number of pips.

Experiment: roll two dice.





Expectation.

How did people do on the midterm?

Distribution.

Summary of distribution?

Average!

Expectation - Definition

Definition: The **expected value** of a random variable X is

$$E[X] = \sum_a a \times Pr[X = a].$$

The expected value is also called the mean.

According to our intuition, we expect that if we repeat an experiment a large number N of times and if $X_1, \ldots, X_N$ are the successive values of the random variable, then

$$\frac{X_1 + \cdots + X_N}{N} \approx E[X].$$

That is indeed the case, in the same way that the fraction of times that $X = x$ approaches $Pr[X = x]$.

This (nontrivial) result is called the Law of Large Numbers.

The subjectivist(bayesian) interpretation of $E[X]$ is less obvious.

Expectation: A Useful Fact

Theorem:

$$E[X] = \sum_{\omega} X(\omega) \times Pr[\omega].$$

Proof:

$$\begin{aligned} E[X] &= \sum_{a} a \times Pr[X = a] \\ &= \sum_{a} a \times \sum_{\omega : X(\omega) = a} Pr[\omega] \\ &= \sum_{a} \sum_{\omega : X(\omega) = a} X(\omega) Pr[\omega] \\ &= \sum_{\omega} X(\omega) Pr[\omega] \end{aligned}$$

$\square$

Distributive property of multiplication over addition.

An Example

Flip a fair coin three times.

$\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.

$X =$ number of H's: $\{3, 2, 2, 2, 1, 1, 1, 0\}$.

Thus,

$$\sum_{\omega} X(\omega) Pr[\omega] = \{3+2+2+2+1+1+1+0\} \times \frac{1}{8}.$$

Also,

$$\sum_{a} a \times Pr[X = a] = 3 \times \frac{1}{8} + 2 \times \frac{3}{8} + 1 \times \frac{3}{8} + 0 \times \frac{1}{8}.$$

What's the answer? Uh.... $\frac{3}{2}$

Expectation and Average.

There are n students in the class;

$X(m)$ = score of student m, for $m = 1, 2, \ldots, n$.

"Average score" of the n students: add scores and divide by n:

$$\text{Average} = \frac{X(1) + X(1) + \cdots + X(n)}{n}.$$

Experiment: choose a student uniformly at random.
Uniform sample space: $\Omega = \{1, 2, \cdots, n\}, Pr[\omega] = 1/n$, for all ω.
Random Variable: midterm score: $X(\omega)$.
Expectation:

$$E(X) = \sum_{\omega} X(\omega) Pr[\omega] = \sum_{\omega} X(\omega) \frac{1}{n}.$$

Hence,

$$\text{Average } = E(X).$$

This holds for a uniform probability space.

Named Distributions.

Some distributions come up over and over again.

...like “choose” or “stars and bars”....

Let’s cover some.

The binomial distribution.

Flip n coins with heads probability p.

Random variable: number of heads.

Binomial Distribution: $Pr[X = i]$, for each i.

How many sample points in event "$X = i$"?
i heads out of n coin flips $\implies \binom{n}{i}$

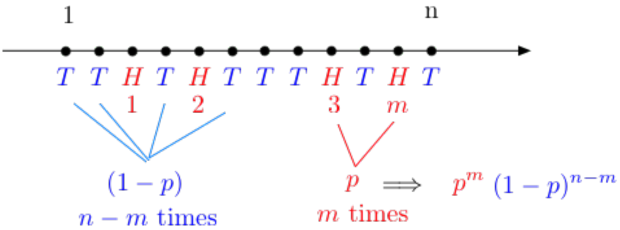
What is the probability of ω if ω has i heads?
Probability of heads in any position is p.
Probability of tails in any position is $(1-p)$.
So, we get

$$Pr[\omega] = p^i(1-p)^{n-i}.$$

Probability of "$X = i$" is sum of $Pr[\omega]$, $\omega \in$ "$X = i$".

$$Pr[X = i] = \binom{n}{i} p^i (1-p)^{n-i}, i = 0, 1, \ldots, n : B(n,p) \text{ distribution}$$

The binomial distribution.



$\binom{n}{m}$ outcomes with m Hs and $n-m$ Ts

$$\Longrightarrow \boxed{Pr[X=m] = \binom{n}{m} p^m (1-p)^{n-m}}$$

Error channel and...

A packet is corrupted with probability p.

Send $n+2k$ packets.

Probability of at most k corruptions.

$$\sum_{i \leq k} \binom{n+2k}{i} p^i (1-p)^{n+2k-i}.$$

Also distribution in polling, experiments, etc.

Expectation of Binomial Distibution

Parameter p and n. What is expectation?

$$Pr[X = i] = \binom{n}{i} p^i (1-p)^{n-i}, i = 0, 1, \ldots, n : B(n, p) \text{ distribution}$$

$$E[X] = \sum_i i \times Pr[X = i].$$

Uh oh? Well... It is pn.

Proof? After linearity of expectation this is easy.

Waiting is good.

Uniform Distribution

Roll a six-sided balanced die. Let X be the number of pips (dots). Then X is equally likely to take any of the values $\{1, 2, \ldots, 6\}$. We say that X is *uniformly distributed* in $\{1, 2, \ldots, 6\}$.

More generally, we say that X is uniformly distributed in $\{1, 2, \ldots, n\}$ if $Pr[X = m] = 1/n$ for $m = 1, 2, \ldots, n$.
In that case,

$$E[X] = \sum_{m=1}^{n} mPr[X = m] = \sum_{m=1}^{n} m \times \frac{1}{n} = \frac{1}{n} \frac{n(n+1)}{2} = \frac{n+1}{2}.$$

Geometric Distribution

Let's flip a coin with $Pr[H] = p$ until we get H.

For instance:

$$\begin{aligned}
\omega_1 &= H, \text{ or} \\
\omega_2 &= T\ H, \text{ or} \\
\omega_3 &= T\ T\ H, \text{ or} \\
\omega_n &= T\ T\ T\ T\ \cdots\ T\ H.
\end{aligned}$$

Note that $\Omega = \{\omega_n, n = 1, 2, \ldots\}$.

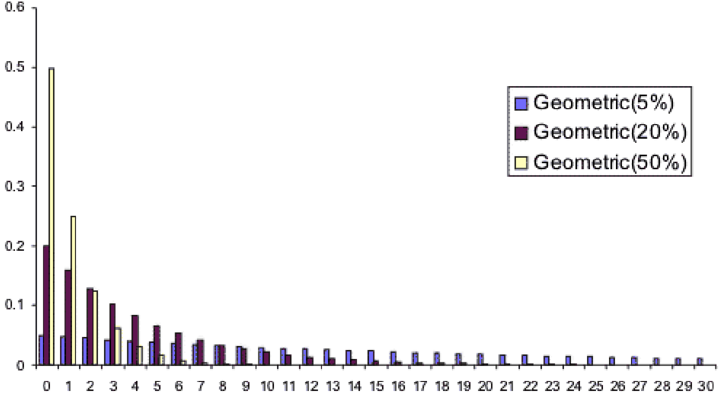
Let X be the number of flips until the first H. Then, $X(\omega_n) = n$.

Also,

$$Pr[X = n] = (1-p)^{n-1}p, \ n \geq 1.$$

Geometric Distribution

$$Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$



Geometric Distribution

$$Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$

Note that

$$\sum_{n=1}^{\infty} Pr[X_n] = \sum_{n=1}^{\infty} (1-p)^{n-1}p = p\sum_{n=1}^{\infty}(1-p)^{n-1} = p\sum_{n=0}^{\infty}(1-p)^{n}.$$

Now, if $|a| < 1$, then $S := \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$. Indeed,

$$\begin{aligned} S &= 1 + a + a^2 + a^3 + \cdots \\ aS &= \phantom{1 + {}} a + a^2 + a^3 + a^4 + \cdots \\ (1-a)S &= 1 + a - a + a^2 - a^2 + \cdots = 1. \end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} Pr[X_n] = p\,\frac{1}{1-(1-p)} = 1.$$

Geometric Distribution: Expectation

$$X =_D G(p), \text{ i.e., } Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$

One has

$$E[X] = \sum_{n=1}^{\infty} nPr[X = n] = \sum_{n=1}^{\infty} n(1-p)^{n-1}p.$$

Thus,

$$\begin{aligned}
E[X] &= p + 2(1-p)p + 3(1-p)^2 p + 4(1-p)^3 p + \cdots \\
(1-p)E[X] &= \qquad\quad (1-p)p + 2(1-p)^2 p + 3(1-p)^3 p + \cdots \\
pE[X] &= p + \ (1-p)p + \ (1-p)^2 p + \ (1-p)^3 p + \cdots \\
&\qquad\qquad \text{by subtracting the previous two identities} \\
&= \sum_{n=1}^{\infty} Pr[X = n] = 1.
\end{aligned}$$

Hence,

$$E[X] = \frac{1}{p}.$$

Summary

Random Variables

- A random variable X is a function $X : \Omega \to \Re$.
- $Pr[X = a] := Pr[X^{-1}(a)] = Pr[\{\omega \mid X(\omega) = a\}]$.
- $Pr[X \in A] := Pr[X^{-1}(A)]$.
- The distribution of X is the list of possible values and their probability: $\{(a, Pr[X = a]), a \in \mathscr{A}\}$.
- $E[X] := \sum_a aPr[X = a]$.
- Expectation is Linear.
- $B(n,p), U[1 : n], G(p), P(\lambda)$.