Today

Random Variables.

Quick Review: Probability. Some Rules.

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  - Inclusion/Exclusion: $Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B]$.
  - Simple Total Probability: $Pr[B] = Pr[A \cap B] + Pr[\overline{A} \cap B]$.
  - Complement: $Pr[\overline{A}] = 1 - Pr[A]$.
  - Union Bound. Total Probability.
- **Conditional Probability:** $Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}$
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1. Random Variables.
2. Expectation
3. Distributions.

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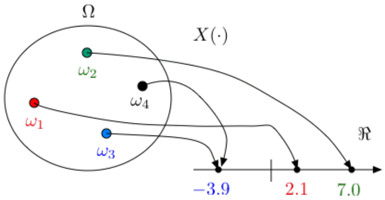

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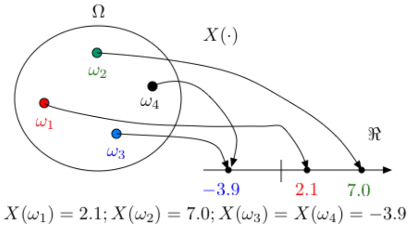

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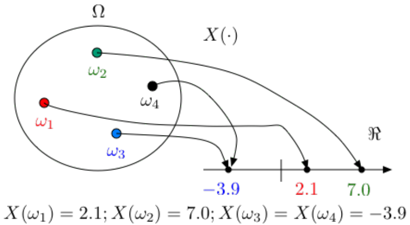

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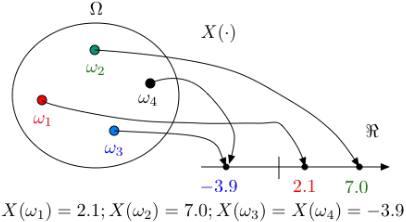

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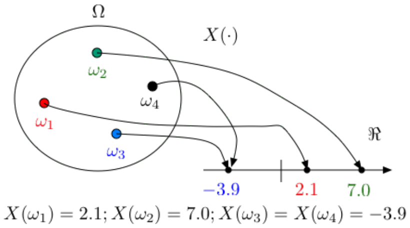

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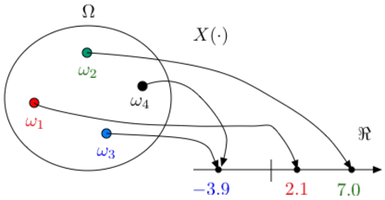

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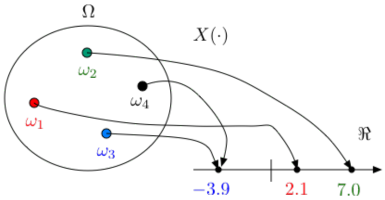

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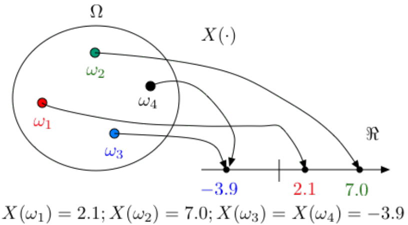

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Note:Random variable induces partition:
$A_y = \{\omega \in \Omega : X(\omega) = y\} = X^{-1}(y)$

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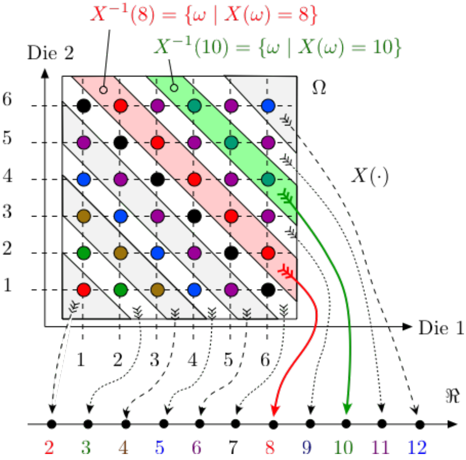

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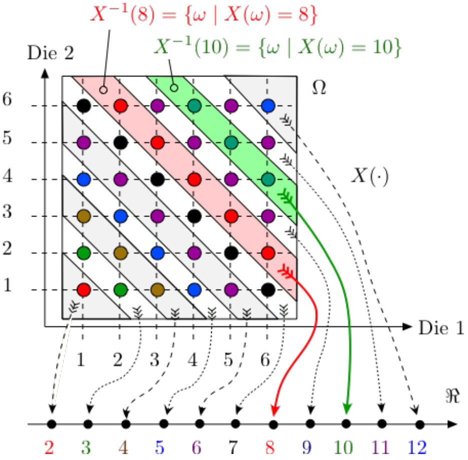

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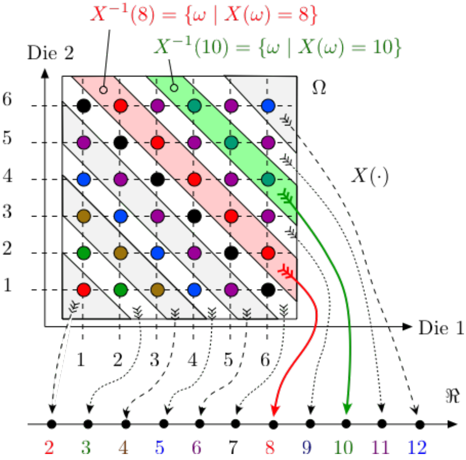

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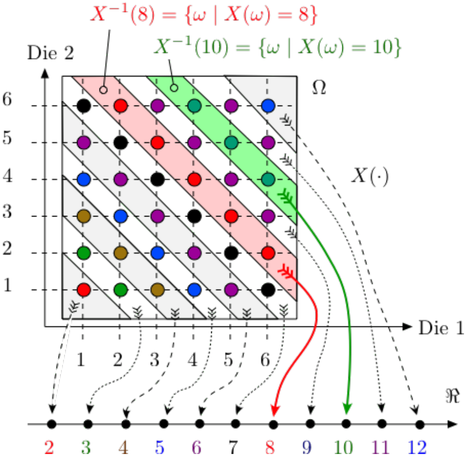

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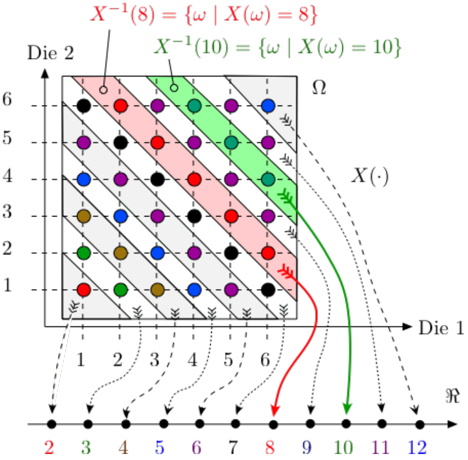

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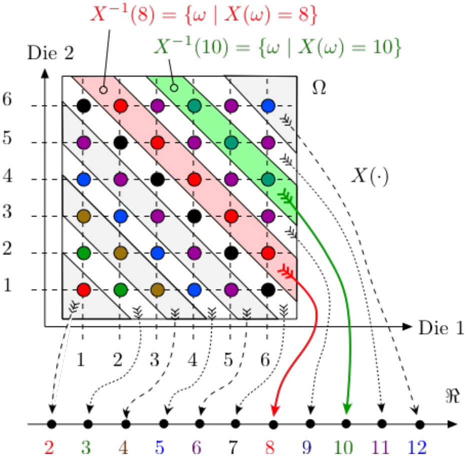

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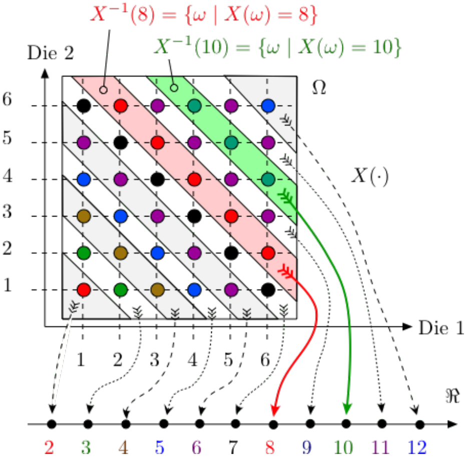

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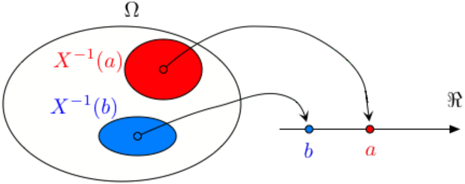

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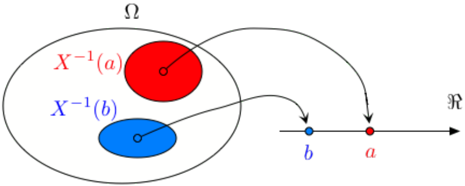

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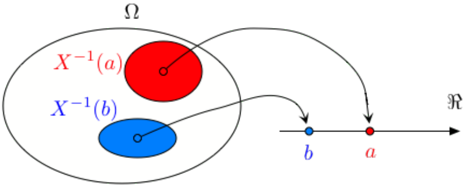

$$Pr[X = a] := Pr[X^{-1}(a)] \text{ where } X^{-1}(a) := \{\omega \mid X(\omega) = a\}.$$

Handing back assignments

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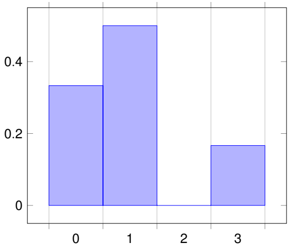

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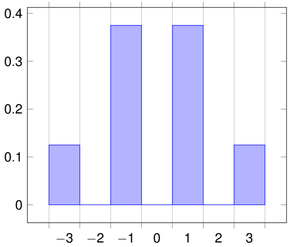

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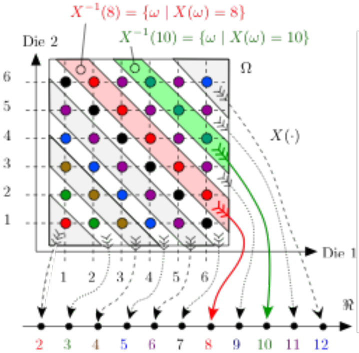

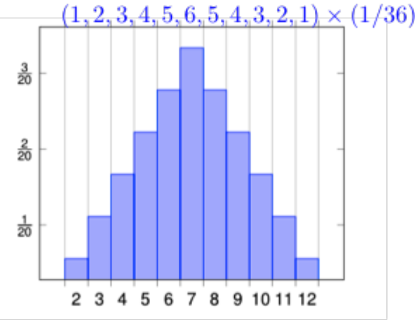

Expectation.

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Distribution.

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Summary of distribution?

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The subjectivist(bayesian) interpretation of $E[X]$ is less obvious.

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Distributive property of multiplication over addition.

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This holds for a uniform probability space.

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Let’s cover some.

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Flip n coins with heads probability p.

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What is the probability of ω if ω has i heads?
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The binomial distribution.

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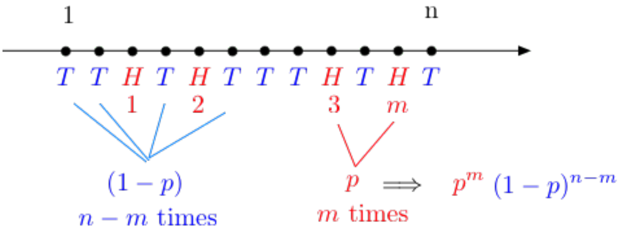

$\binom{n}{m}$ outcomes with m Hs and $n-m$ Ts

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Also distribution in polling, experiments, etc.

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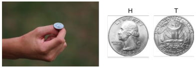

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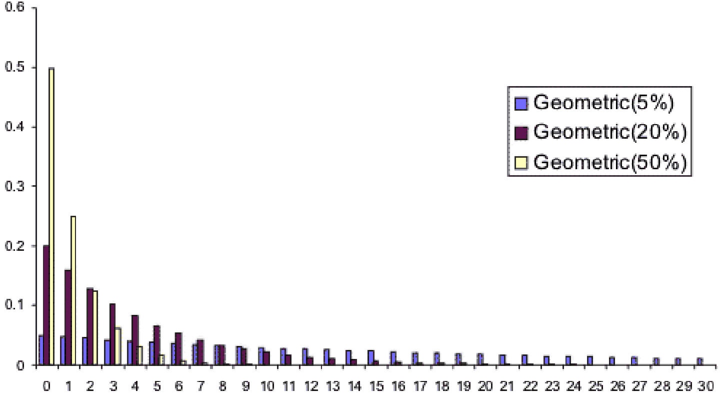

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- The distribution of X is the list of possible values and their probability: $\{(a, Pr[X = a]), a \in \mathscr{A}\}$.
- $E[X] := \sum_a aPr[X = a]$.
- Expectation is Linear.

Summary

Random Variables

- A random variable X is a function $X : \Omega \to \Re$.
- $Pr[X = a] := Pr[X^{-1}(a)] = Pr[\{\omega \mid X(\omega) = a\}]$.
- $Pr[X \in A] := Pr[X^{-1}(A)]$.
- The distribution of X is the list of possible values and their probability: $\{(a, Pr[X = a]), a \in \mathscr{A}\}$.
- $E[X] := \sum_a aPr[X = a]$.
- Expectation is Linear.
- $B(n,p), U[1:n], G(p), P(\lambda)$.