CS70.

1. Random Variables: Brief Review
2. Joint Distributions.
3. Linearity of Expectation

Random Variables: Definitions

Definition
A random variable, X, for a random experiment with sample space Ω is a function $X : \Omega \to \Re$.

Thus, $X(\cdot)$ assigns a real number $X(\omega)$ to each $\omega \in \Omega$.

Definitions
(a) For $a \in \Re$, one defines

$$X^{-1}(a) := \{\omega \in \Omega \mid X(\omega) = a\}.$$

(b) For $A \subset \Re$, one defines

$$X^{-1}(A) := \{\omega \in \Omega \mid X(\omega) \in A\}.$$

(c) The probability that $X = a$ is defined as

$$Pr[X = a] = Pr[X^{-1}(a)].$$

(d) The probability that $X \in A$ is defined as

$$Pr[X \in A] = Pr[X^{-1}(A)].$$

(e) The distribution of a random variable X, is

$$\{(a, Pr[X = a]) : a \in \mathscr{A}\},$$

where $\mathscr{A}$ is the *range* of X. That is, $\mathscr{A} = \{X(\omega), \omega \in \Omega\}$.

Expectation - Definition

Definition: The **expected value** (or mean, or expectation) of a random variable X is

$$E[X] = \sum_a a \times Pr[X = a].$$

Theorem:

$$E[X] = \sum_\omega X(\omega) \times Pr[\omega].$$

Proof:

$$\begin{aligned} E[X] &= \sum_a a \times Pr[X = a] \\ &= \sum_a a \times \sum_{\omega : X(\omega) = a} Pr[\omega] \\ &= \sum_a \sum_{\omega : X(\omega) = a} X(\omega) Pr[\omega] \\ &= \sum_\omega X(\omega) Pr[\omega] \end{aligned}$$

□

An Example

Flip a fair coin three times.

$\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.

$X =$ number of H's: $\{3, 2, 2, 2, 1, 1, 1, 0\}$.

Thus,

$$\sum_\omega X(\omega) Pr[\omega] = \{3 + 2 + 2 + 2 + 1 + 1 + 1 + 0\} \times \frac{1}{8}.$$

Also,

$$\sum_a a \times Pr[X = a] = 3 \times \frac{1}{8} + 2 \times \frac{3}{8} + 1 \times \frac{3}{8} + 0 \times \frac{1}{8}.$$

Win or Lose.

Expected winnings for heads/tails games, with 3 flips?
Recall the definition of the random variable X:
$\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\} \to \{3, 1, 1, -1, 1, -1, -1, -3\}$.

$$E[X] = 3 \times \frac{1}{8} + 1 \times \frac{3}{8} - 1 \times \frac{3}{8} - 3 \times \frac{1}{8} = 0.$$

Can you ever win 0?

Apparently: expected value is not a common value, by any means.

The expected value of X is not the value that you expect!
It is the average value per experiment, if you perform the experiment many times:

$$\frac{X_1 + \cdots + X_n}{n}, \text{ when } n \gg 1.$$

The fact that this average converges to $E[X]$ is a theorem:
the Law of Large Numbers. (See later.)

Multiple Random Variables.

Experiment: toss two coins. $\Omega = \{HH, TH, HT, TT\}$.

$$X_1(\omega) = \begin{cases} 1, & \text{if coin 1 is heads} \\ 0, & \text{otherwise} \end{cases} \qquad X_2(\omega) = \begin{cases} 1, & \text{if coin 2 is heads} \\ 0, & \text{otherwise} \end{cases}$$

Multiple Random Variables: setup.

Joint Distribution: $\{(a,b,Pr[X=a,Y=b]) : a\in\mathscr{A}, b\in\mathscr{B}\}$, where $\mathscr{A}$ ($\mathscr{B}$) is possible values of X (Y).

$$\sum_{a\in\mathscr{A},b\in\mathscr{B}} Pr[X=a,Y=b]=1$$

Marginal for X: $Pr[X=a]=\sum_{b\in\mathscr{B}} Pr[X=a,Y=b]$.
Marginal for Y: $Pr[Y=b]=\sum_{a\in\mathscr{A}} Pr[X=a,Y=b]$.

X/Y	1	2	3	X
1	.2	.1	.1	.4
2	0	0	.3	.3
3	.1	0	.2	.3
Y	.3	.1	.2 .6	

Conditional Probability: $Pr[X=a|Y=b]=\frac{Pr[X=a,Y=b]}{Pr[Y=b]}$.

Review: Independence of Events

- Events A,B are independent if $Pr[A\cap B]=Pr[A]Pr[B]$.
- Events A,B,C are mutually independent if

  A,B are independent, A,C are independent, B,C are independent

  and $Pr[A\cap B\cap C]=Pr[A]Pr[B]Pr[C]$.
- Events $\{A_n, n\geq 0\}$ are mutually independent if ....
- Example: $X,Y\in\{0,1\}$ two fair coin flips $\Rightarrow X,Y,X\oplus Y$ are pairwise independent but not mutually independent.
- Example: $X,Y,Z\in\{0,1\}$ three fair coin flips are mutually independent.

Independent Random Variables.

Definition: Independence

The random variables X and Y are **independent** if and only if

$$Pr[Y=b|X=a]=Pr[Y=b], \text{ for all } a \text{ and } b.$$

Fact:

X,Y are independent if and only if

$$Pr[X=a,Y=b]=Pr[X=a]Pr[Y=b], \text{ for all } a \text{ and } b.$$

Follows from $Pr[A\cap B]=Pr[A|B]Pr[B]$ (Product rule.)

Independence: Examples

Example 1
Roll two die. X,Y = number of pips on the two dice. X,Y are independent.

Indeed: $Pr[X=a,Y=b]=\frac{1}{36}, Pr[X=a]=Pr[Y=b]=\frac{1}{6}$.

Example 2
Roll two die. X = total number of pips, Y = number of pips on die 1 minus number on die 2. X and Y are not independent.

Indeed: $Pr[X=12,Y=1]=0\neq Pr[X=12]Pr[Y=1]>0$.

Example 3
Flip a fair coin five times, X = number of Hs in first three flips, Y = number of Hs in last two flips. X and Y are independent.

Indeed:

$$Pr[X=a,Y=b]=\binom{3}{a}\binom{2}{b}2^{-5}=\binom{3}{a}2^{-3}\times\binom{2}{b}2^{-2}=Pr[X=a]Pr[Y=b].$$

Linearity of Expectation

Theorem:
$E[X+Y]=E[X]+E[Y]$
$E[cX]=cE[X]$

Proof: $E[X]=\sum_{\omega\in\Omega}X(\omega)\times P[\omega]$.

$$\begin{aligned} E[X+Y] &= \sum_{\omega\in\Omega}(X(\omega)+Y(\omega))Pr[\omega] \\ &= \sum_{\omega\in\Omega}X(\omega)Pr[\omega]+Y(\omega)Pr[\omega] \\ &= \sum_{\omega\in\Omega}X(\omega)Pr[\omega]+\sum_{\omega\in\Omega}Y(\omega)Pr[\omega] \\ &= E[X]+E[Y] \end{aligned}$$

Indicators

Definition
Let A be an event. The random variable X defined by

$$X(\omega)=\begin{cases}1, & \text{if } \omega\in A\\ 0, & \text{if } \omega\notin A\end{cases}$$

is called the indicator of the event A.

Note that $Pr[X=1]=Pr[A]$ and $Pr[X=0]=1-Pr[A]$.

Hence,
$$E[X]=1\times Pr[X=1]+0\times Pr[X=0]=Pr[A].$$

This random variable $X(\omega)$ is sometimes written as

$$1\{\omega\in A\} \text{ or } 1_A(\omega).$$

Thus, we will write $X=1_A$.

Linearity of Expectation

Theorem: Expectation is linear

$$E[a_1X_1+\cdots+a_nX_n]=a_1E[X_1]+\cdots+a_nE[X_n].$$

Proof:

$$\begin{aligned}&E[a_1X_1+\cdots+a_nX_n]\\&=\sum_\omega(a_1X_1+\cdots+a_nX_n)(\omega)Pr[\omega]\\&=\sum_\omega(a_1X_1(\omega)+\cdots+a_nX_n(\omega))Pr[\omega]\\&=a_1\sum_\omega X_1(\omega)Pr[\omega]+\cdots+a_n\sum_\omega X_n(\omega)Pr[\omega]\\&=a_1E[X_1]+\cdots+a_nE[X_n].\end{aligned}$$

□

Note: If we had defined $Y=a_1X_1+\cdots+a_nX_n$ has had tried to compute $E[Y]=\sum_y yPr[Y=y]$, we would have been in trouble!

Using Linearity - 1: Pips (dots) on dice

Roll a die n times.

X_m = number of pips on roll m.

$X=X_1+\cdots+X_n$ = total number of pips in n rolls.

$$\begin{aligned}E[X]&=E[X_1+\cdots+X_n]\\&=E[X_1]+\cdots+E[X_n],\text{ by linearity}\\&=nE[X_1],\text{ because the }X_m\text{ have the same distribution}\end{aligned}$$

Now,

$$E[X_1]=1\times\frac{1}{6}+\cdots+6\times\frac{1}{6}=\frac{6\times7}{2}\times\frac{1}{6}=\frac{7}{2}.$$

Hence,

$$E[X]=\frac{7n}{2}.$$

Note: Computing $\sum_x xPr[X=x]$ directly is not easy!

Using Linearity - 2: Fixed point.

Hand out assignments at random to n students.

X = number of students that get their own assignment back.

$X=X_1+\cdots+X_n$ where
$X_m=1\{$student m gets his/her own assignment back$\}$.

One has

$$\begin{aligned}E[X]&=E[X_1+\cdots+X_n]\\&=E[X_1]+\cdots+E[X_n],\text{ by linearity}\\&=nE[X_1],\text{ because all the }X_m\text{ have the same distribution}\\&=nPr[X_1=1],\text{ because }X_1\text{ is an indicator}\\&=n(1/n),\text{ because student 1 is equally likely}\\&\qquad\text{to get any one of the }n\text{ assignments}\\&=1.\end{aligned}$$

Note that linearity holds even though the X_m are not independent (whatever that means).

Note: What is $Pr[X=m]$? Tricky ....

Using Linearity - 3: Binomial Distribution.

Flip n coins with heads probability p. X - number of heads

Binomial Distibution: $Pr[X=i]$, for each i.

$$Pr[X=i]=\binom{n}{i}p^i(1-p)^{n-i}.$$

$$E[X]=\sum_i i\times Pr[X=i]=\sum_i i\times\binom{n}{i}p^i(1-p)^{n-i}.$$

Uh oh. ... Or... a better approach: Let

$$X_i=\begin{cases}1 & \text{if } i\text{th flip is heads}\\ 0 & \text{otherwise}\end{cases}$$

$E[X_i]=1\times Pr[\text{"heads"}]+0\times Pr[\text{"tails"}]=p.$

Moreover $X=X_1+\cdots X_n$ and

$E[X]=E[X_1]+E[X_2]+\cdots E[X_n]=n\times E[X_i]=np.$

Using Linearity - 4

Assume A and B are disjoint events. Then $1_{A\cup B}(\omega)=1_A(\omega)+1_B(\omega)$.

Taking expectation, we get

$$Pr[A\cup B]=E[1_{A\cup B}]=E[1_A+1_B]=E[1_A]+E[1_B]=Pr[A]+Pr[B].$$

In general, $1_{A\cup B}(\omega)=1_A(\omega)+1_B(\omega)-1_{A\cap B}(\omega)$.

Taking expectation, we get $Pr[A\cup B]=Pr[A]+Pr[B]-Pr[A\cap B]$.

Observe that if $Y(\omega)=b$ for all ω, then $E[Y]=b$.

Thus, $E[X+b]=E[X]+b$.

Empty Bins

Experiment: Throw m balls into n bins.

Y - number of empty bins.

Distribution is horrible.

Expectation? X_i - indicator for bin i being empty.

$Y=X_1+\cdots X_n$.

$Pr[X_1=1]=(1-\frac{1}{n})^m$. $\rightarrow$ $E[Y]=n(1-\frac{1}{n})^m$.

For $n=m$ and large n, $(1-1/n)^n\approx\frac{1}{e}$.

$\frac{n}{e}\approx 0.368n$ empty bins on average.

Coupon Collectors Problem.

Experiment: Get random coupon from n until get all n coupons.
Outcomes: $\{123145...,56765...\}$
Random Variable: X - length of outcome.

Today: $E[X]$?

Geometric Distribution: Expectation

$$X =_D G(p), \text{ i.e., } Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$

One has

$$E[X] = \sum_{n=1}^{\infty} nPr[X = n] = \sum_{n=1}^{\infty} n(1-p)^{n-1}p.$$

Thus,

$$\begin{aligned} E[X] &= p + 2(1-p)p + 3(1-p)^2p + 4(1-p)^3p + \cdots \\ (1-p)E[X] &= \quad (1-p)p + 2(1-p)^2p + 3(1-p)^3p + \cdots \\ pE[X] &= p + \quad (1-p)p + \quad (1-p)^2p + \quad (1-p)^3p + \cdots \end{aligned}$$

by subtracting the previous two identities

$$= \sum_{n=1}^{\infty} Pr[X = n] = 1.$$

Hence,

$$E[X] = \frac{1}{p}.$$

Time to collect coupons

X-time to get n coupons.

X_1 - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

X_2 - time to get second coupon after getting first.

Pr["get second coupon"|"got ~~milk~~ first coupon"] $= \frac{n-1}{n}$

$E[X_2]$? Geometric ! ! ! $\implies E[X_2] = \frac{1}{p} = \frac{1}{\frac{n-1}{n}} = \frac{n}{n-1}$.

Pr["getting ith coupon|"got $i-1$rst coupons"] $= \frac{n-(i-1)}{n} = \frac{n-i+1}{n}$

$E[X_i] = \frac{1}{p} = \frac{n}{n-i+1}, i = 1, 2, \ldots, n.$

$$\begin{aligned} E[X] &= E[X_1] + \cdots + E[X_n] = \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{1} \\ &= n(1 + \frac{1}{2} + \cdots + \frac{1}{n}) =: nH(n) \approx n(\ln n + \gamma) \end{aligned}$$

Review: Harmonic sum

$$H(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \int_1^n \frac{1}{x} dx = \ln(n).$$

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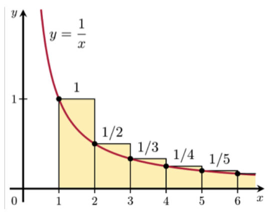
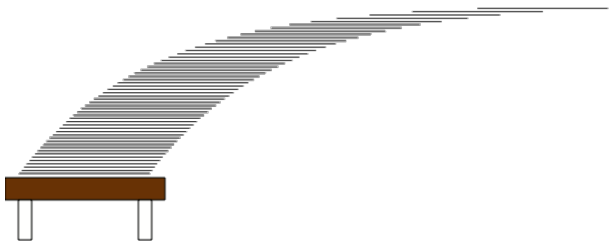


A good approximation is

$H(n) \approx \ln(n) + \gamma$ where $\gamma \approx 0.58$ (Euler-Mascheroni constant).

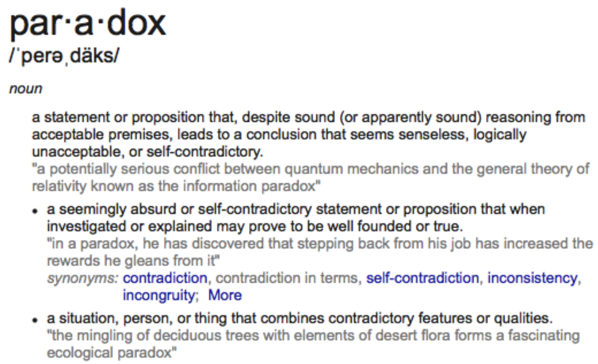
Harmonic sum: Paradox

Consider this stack of cards (no glue!):



If each card has length 2, the stack can extend $H(n)$ to the right of the table. As n increases, you can go as far as you want!

Paradox

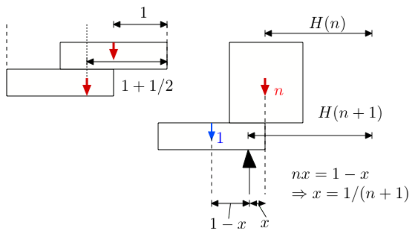


par·a·dox
/ˈperəˌdäks/
noun

a statement or proposition that, despite sound (or apparently sound) reasoning from acceptable premises, leads to a conclusion that seems senseless, logically unacceptable, or self-contradictory.
"a potentially serious conflict between quantum mechanics and the general theory of relativity known as the information paradox"

- a seemingly absurd or self-contradictory statement or proposition that when investigated or explained may prove to be well founded or true.
"in a paradox, he has discovered that stepping back from his job has increased the rewards he gleans from it"
synonyms: contradiction, contradiction in terms, self-contradiction, inconsistency, incongruity; More
- a situation, person, or thing that combines contradictory features or qualities.
"the mingling of deciduous trees with elements of desert flora forms a fascinating ecological paradox"

Stacking



The cards have width 2. Induction shows that the center of gravity after n cards is $H(n)$ away from the right-most edge.
Video.

Summary

Probability Space: Ω, $Pr[\omega] \geq 0$, $\sum_\omega Pr[\omega] = 1$.

Random Variable: Function on Sample Space.

Distribution: Function $Pr[X = a] \geq 0$. $\sum_a Pr[X = a] = 1$.

Expectation: $E[X] = \sum_\omega Pr[\omega] = \sum_a Pr[X = a]$.

Many Random Variables: each one function on a sample space.

Joint Distributions: Function $Pr[X = a, Y = b] \geq 0$.
$\sum_{a,b} Pr[X = a, Y = b] = 1$.

Linearity of Expectation: $E[X+Y] = E[X] + E[Y]$.

Applications: compute expectations by decomposing.

  Indicators: Empty bins, Fixed points.
  Time to Coupon: Sum times to "next" coupon.

$Y = f(X)$ is Random Variable.
 Distribution of Y from distribution of X.

Calculating $E[g(X)]$

Let $Y = g(X)$. Assume that we know the distribution of X.

We want to calculate $E[Y]$.

Method 1: We calculate the distribution of Y:

$$Pr[Y = y] = Pr[X \in g^{-1}(y)] \text{ where } g^{-1}(x) = \{x \in \Re : g(x) = y\}.$$

This is typically rather tedious!

Method 2: We use the following result.

Theorem:

$$E[g(X)] = \sum_x g(x) Pr[X = x].$$

Proof:

$$\begin{aligned} E[g(X)] &= \sum_\omega g(X(\omega)) Pr[\omega] = \sum_x \sum_{\omega \in X^{-1}(x)} g(X(\omega)) Pr[\omega] \\ &= \sum_x \sum_{\omega \in X^{-1}(x)} g(x) Pr[\omega] = \sum_x g(x) \sum_{\omega \in X^{-1}(x)} Pr[\omega] \\ &= \sum_x g(x) Pr[X = x]. \end{aligned}$$

□

An Example

Let X be uniform in $\{-2,-1,0,1,2,3\}$.

Let also $g(X) = X^2$. Then (method 2)

$$\begin{aligned} E[g(X)] &= \sum_{x=-2}^{3} x^2 \frac{1}{6} \\ &= \{4+1+0+1+4+9\}\frac{1}{6} = \frac{19}{6}. \end{aligned}$$

Method 1 - We find the distribution of $Y = X^2$:

$$Y = \begin{cases} 4, & \text{w.p. } \frac{2}{6} \\ 1, & \text{w.p. } \frac{2}{6} \\ 0, & \text{w.p. } \frac{1}{6} \\ 9, & \text{w.p. } \frac{1}{6}. \end{cases}$$

Thus,

$$E[Y] = 4\frac{2}{6} + 1\frac{2}{6} + 0\frac{1}{6} + 9\frac{1}{6} = \frac{19}{6}.$$