CS70

Another Distribution:Poisson

Variance/ Covariance.

Poisson: Motivation and derivation.

McDonalds: How many person arrive in an hour?

Know: average is λ.

What is distribution?

Example: $Pr[2\lambda$ arrivals $]$?

Assumption: "arrivals are independent."

Derivation: cut hour into n intervals of length $1/n$.
$Pr[$ two arrivals $]$ is "$(\lambda/n)^2$" or small if n is large.
Model with binomial.

Poisson

Experiment: flip a coin n times. The coin is such that $Pr[H] = \lambda/n$.
Random Variable: X - number of heads. Thus, $X = B(n, \lambda/n)$.
Poisson Distribution is distribution of X "for large n."
We expect $X \ll n$. For $m \ll n$ one has

$$\begin{aligned}
Pr[X = m] &= \binom{n}{m} p^m (1-p)^{n-m}, \text{ with } p = \lambda/n \\
&= \frac{n(n-1)\cdots(n-m+1)}{m!} \left(\frac{\lambda}{n}\right)^m \left(1 - \frac{\lambda}{n}\right)^{n-m} \\
&= \frac{n(n-1)\cdots(n-m+1)}{n^m} \frac{\lambda^m}{m!} \left(1 - \frac{\lambda}{n}\right)^{n-m} \\
&\approx^{(1)} \frac{\lambda^m}{m!} \left(1 - \frac{\lambda}{n}\right)^{n-m} \approx^{(2)} \frac{\lambda^m}{m!} \left(1 - \frac{\lambda}{n}\right)^{n} \approx \frac{\lambda^m}{m!} e^{-\lambda}.
\end{aligned}$$

For (1) we used $m \ll n$; for (2) we used $(1 - a/n)^n \approx e^{-a}$.

Poisson Distribution: Definition and Mean

Definition Poisson Distribution with parameter $\lambda > 0$

$$X = P(\lambda) \Leftrightarrow Pr[X = m] = \frac{\lambda^m}{m!} e^{-\lambda}, m \geq 0.$$

Fact: $E[X] = \lambda$.

Proof:

$$\begin{aligned}
E[X] &= \sum_{m=1}^{\infty} m \times \frac{\lambda^m}{m!} e^{-\lambda} = e^{-\lambda} \sum_{m=1}^{\infty} \frac{\lambda^m}{(m-1)!} \\
&= e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^{m+1}}{m!} = e^{-\lambda} \lambda \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \\
&= e^{-\lambda} \lambda e^{\lambda} = \lambda.
\end{aligned}$$

$\square$

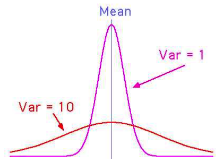
Simeon Poisson

The Poisson distribution is named after:

Siméon Poisson

Siméon Denis Poisson (1781–1840)

Variance



The variance measures the deviation from the mean value.

Definition: The variance of X is

$$\sigma^2(X) := var[X] = E[(X - E[X])^2].$$

$\sigma(X)$ is called the standard deviation of X.

Variance and Standard Deviation

Fact:

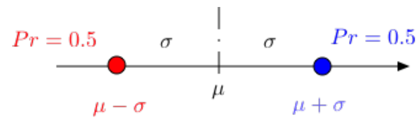
$$var[X] = E[X^2] - E[X]^2.$$

Indeed:

$$\begin{aligned} var(X) &= E[(X - E[X])^2] \\ &= E[X^2 - 2XE[X] + E[X]^2) \\ &= E[X^2] - 2E[X]E[X] + E[X]^2, \text{ by linearity} \\ &= E[X^2] - E[X]^2. \end{aligned}$$

A simple example

This example illustrates the term 'standard deviation.'



Consider the random variable X such that

$$X = \begin{cases} \mu - \sigma, & \text{w.p. } 1/2 \\ \mu + \sigma, & \text{w.p. } 1/2. \end{cases}$$

Then, $E[X] = \mu$ and $(X - E[X])^2 = \sigma^2$. Hence,

$$var(X) = \sigma^2 \text{ and } \sigma(X) = \sigma.$$

Example

Consider X with

$$X = \begin{cases} -1, & \text{w. p. } 0.99 \\ 99, & \text{w. p. } 0.01. \end{cases}$$

Then

$$\begin{aligned} E[X] &= -1 \times 0.99 + 99 \times 0.01 = 0. \\ E[X^2] &= 1 \times 0.99 + (99)^2 \times 0.01 \approx 100. \\ Var(X) &\approx 100 \implies \sigma(X) \approx 10. \end{aligned}$$

Also,

$$E(|X|) = 1 \times 0.99 + 99 \times 0.01 = 1.98.$$

Thus, $\sigma(X) = \sqrt{E[(X - E(X))^2]} \neq E[|X - E[X]|]$!

Exercise: How big can you make $\frac{\sigma(X)}{E[|X - E[X]|]}$?

Uniform

Assume that $Pr[X = i] = 1/n$ for $i \in \{1, \ldots, n\}$. Then

$$\begin{aligned} E[X] &= \sum_{i=1}^{n} i \times Pr[X = i] = \frac{1}{n} \sum_{i=1}^{n} i \\ &= \frac{1}{n} \frac{n(n+1)}{2} = \frac{n+1}{2}. \end{aligned}$$

Also,

$$\begin{aligned} E[X^2] &= \sum_{i=1}^{n} i^2 Pr[X = i] = \frac{1}{n} \sum_{i=1}^{n} i^2 \\ &= \frac{1}{n} \frac{(n)(n+1)(n+2)}{6} = \frac{1 + 3n + 2n^2}{6}, \text{ as you can verify.} \end{aligned}$$

This gives

$$var(X) = \frac{1 + 3n + 2n^2}{6} - \frac{(n+1)^2}{4} = \frac{n^2 - 1}{12}.$$

(Sort of $\int_0^{1/2} x^2 dx = \frac{x^3}{3}$.)

Variance of geometric distribution.

X is a geometrically distributed RV with parameter p.
Thus, $Pr[X = n] = (1 - p)^{n-1} p$ for $n \geq 1$. Recall $E[X] = 1/p$.

$$\begin{aligned} E[X^2] &= p + 4p(1-p) + 9p(1-p)^2 + \ldots \\ -(1-p)E[X^2] &= \quad -[p(1-p) + 4p(1-p)^2 + \ldots] \\ pE[X^2] &= p + 3p(1-p) + 5p(1-p)^2 + \ldots \\ &= 2(p + 2p(1-p) + 3p(1-p)^2 + ..) \quad E[X]! \\ &\quad -(p + p(1-p) + p(1-p)^2 + \ldots) \quad \text{Distribution.} \\ pE[X^2] &= 2E[X] - 1 \\ &= 2(\frac{1}{p}) - 1 = \frac{2-p}{p} \end{aligned}$$

$\implies E[X^2] = (2 - p)/p^2$ and
$var[X] = E[X^2] - E[X]^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$.
$\sigma(X) = \frac{\sqrt{1-p}}{p} \approx E[X]$ when p is small(ish).

Fixed points.

Number of fixed points in a random permutation of n items.
"Number of student that get homework back."

$X = X_1 + X_2 \cdots + X_n$

where X_i is indicator variable for ith student getting hw back.

$$\begin{aligned} E(X^2) &= \sum_i E(X_i^2) + \sum_{i \neq j} E(X_i X_j). \\ &= n \times \frac{1}{n} + (n)(n-1) \times \frac{1}{n(n-1)} \\ &= 1 + 1 = 2. \end{aligned}$$

$E(X_i^2) = 1 \times Pr[X_i = 1] + 0 \times Pr[X_i = 0]$
$= \frac{1}{n}$
$E(X_i X_j) = 1 \times Pr[X_i = 1 \cap X_j = 1] + 0 \times Pr[\text{"anything else"}]$
$= 1 \times \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$

$Var(X) = E(X^2) - (E(X))^2 = 2 - 1 = 1.$

Variance: binomial.

$$E[X^2] = \sum_{i=0}^{n} i^2 \binom{n}{i} p^i (1-p)^{n-i}.$$
$$= \text{Really???!!\#\#...}$$

Too hard!

Ok.. fine.
Let's do something else.
Maybe not much easier...but there is a payoff.

Properties of variance.

1. $Var(cX) = c^2 Var(X)$, where c is a constant.
   Scales by c^2.
2. $Var(X+c) = Var(X)$, where c is a constant.
   Shifts center.

Proof:

$$\begin{aligned} Var(cX) &= E((cX)^2) - (E(cX))^2 \\ &= c^2E(X^2) - c^2(E(X))^2 = c^2(E(X^2) - E(X)^2) \\ &= c^2 Var(X) \\ Var(X+c) &= E((X+c-E(X+c))^2) \\ &= E((X+c-E(X)-c)^2) \\ &= E((X-E(X))^2) = Var(X) \end{aligned}$$

□

Independent random variables.

Independent: $P[X=a, Y=b] = Pr[X=a]Pr[Y=b]$

Fact: $E[XY] = E[X]E[Y]$ for independent random variables.

$$\begin{aligned} E[XY] = & \sum_a \sum_b a \times b \times PR[X=a, Y=b] \\ = & \sum_a \sum_b a \times b \times PR[X=a]Pr[Y=b] \\ = & (\sum_a aPr[X=a])(\sum_b bPr[Y=b]) \\ = & E[X]E[Y] \end{aligned}$$

Variance of sum of two independent random variables

Theorem:
If X and Y are independent, then

$$Var(X+Y) = Var(X) + Var(Y).$$

Proof:
Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that $E(X) = 0$ and $E(Y) = 0$.

Then, by independence,

$$E(XY) = E(X)E(Y) = 0.$$

Hence,

$$\begin{aligned} var(X+Y) &= E((X+Y)^2) = E(X^2 + 2XY + Y^2) \\ &= E(X^2) + 2E(XY) + E(Y^2) = E(X^2) + E(Y^2) \\ &= var(X) + var(Y). \end{aligned}$$

Variance of sum of independent random variables

Theorem:
If $X, Y, Z, \ldots$ are pairwise independent, then

$$var(X+Y+Z+\cdots) = var(X) + var(Y) + var(Z) + \cdots.$$

Proof:
Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that $E[X] = E[Y] = \cdots = 0$.

Then, by independence,

$$E[XY] = E[X]E[Y] = 0. \text{ Also, } E[XZ] = E[YZ] = \cdots = 0.$$

Hence,

$$\begin{aligned} var(X+Y+Z+\cdots) &= E((X+Y+Z+\cdots)^2) \\ &= E(X^2 + Y^2 + Z^2 + \cdots + 2XY + 2XZ + 2YZ + \cdots) \\ &= E(X^2) + E(Y^2) + E(Z^2) + \cdots + 0 + \cdots + 0 \\ &= var(X) + var(Y) + var(Z) + \cdots. \end{aligned}$$

□

Variance of Binomial Distribution.

Flip coin with heads probability p.
X- how many heads?

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$E(X_i^2) = 1^2 \times p + 0^2 \times (1-p) = p.$
$Var(X_i) = p - (E(X))^2 = p - p^2 = p(1-p).$

$p = 0 \implies Var(X_i) = 0$
$p = 1 \implies Var(X_i) = 0$

$X = X_1 + X_2 + \ldots X_n.$

X_i and X_j are independent: $Pr[X_i = 1 | X_j = 1] = Pr[X_i = 1].$

$$Var(X) = Var(X_1 + \cdots X_n) = np(1-p).$$

Poisson Distribution: Variance.

Definition Poisson Distribution with parameter $\lambda > 0$

$$X = P(\lambda) \Leftrightarrow Pr[X = m] = \frac{\lambda^m}{m!} e^{-\lambda}, m \geq 0.$$

Mean, Variance?

Ugh.

Recall that Poisson is the limit of the Binomial with $p = \lambda/n$ as $n \to \infty$.

Mean: $pn = \lambda$

Variance: $p(1-p)n = \lambda - \lambda^2/n \to \lambda$.

$E(X^2)$? $Var(X) = E(X^2) - (E(X))^2$ or $E(X^2) = Var(X) + E(X)^2$.

$E(X^2) = \lambda + \lambda^2$.

Covariance

Definition The covariance of X and Y is

$$cov(X, Y) := E[(X - E[X])(Y - E[Y])].$$

Fact

$$cov(X, Y) = E[XY] - E[X]E[Y].$$

Proof:
Think about $E[X] = E[Y] = 0$. Just $E[XY]$. $\square$ish.

For the sake of completeness.

$$E[(X - E[X])(Y - E[Y])] = E[XY - E[X]Y - XE[Y] + E[X]E[Y]]$$
$$= E[XY] - E[X]E[Y] - E[X]E[Y] + E[X]E[Y]$$
$$= E[XY] - E[X]E[Y].$$

Correlation

Definition The correlation of X, Y, $Cor(X, Y)$ is

$$corr(X, Y) : \frac{cov(X, Y)}{\sigma(X)\sigma(Y)}.$$

Theorem: $-1 \leq corr(X, Y) \leq 1$.
Proof: Idea: $(a - b)^2 > 0 \to a^2 + b^2 \geq 2ab$.

Simple case: $E[X] = E[Y] = 0$ and $E[X^2] = E[Y^2] = 1$.

$Cor(X, Y) = E[XY]$.

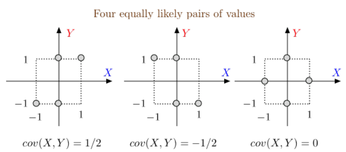
$E[(X - Y)^2] = E[X^2] + E[Y^2] - 2E[XY] = 2(1 - E[XY]) \geq 0$
$\to E[XY] \leq 1$.

$E[(X + Y)^2] = E[X^2] + E[Y^2] + 2E[XY] = 2(1 + E[XY]) \geq 0$
$\to E[XY] \geq -1$.

$\square$

Shifting and scaling doesn't change correlation.

Examples of Covariance



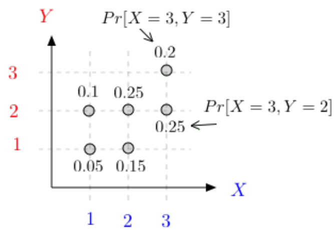
Note that $E[X] = 0$ and $E[Y] = 0$ in these examples. Then $cov(X, Y) = E[XY]$.

When $cov(X, Y) > 0$, the RVs X and Y tend to be large or small together. X and Y are said to be positively correlated.

When $cov(X, Y) < 0$, when X is larger, Y tends to be smaller. X and Y are said to be negatively correlated.

When $cov(X, Y) = 0$, we say that X and Y are uncorrelated.

Examples of Covariance



$E[X] = 1 \times 0.15 + 2 \times 0.4 + 3 \times 0.45 = 2.3$
$E[X^2] = 1^2 \times 0.15 + 2^2 \times 0.4 + 3^2 \times 0.45 = 5.8$
$E[Y] = 1 \times 0.2 + 2 \times 0.6 + 3 \times 0.2 = 2$
$E[Y^2] = 1 \times 0.2 + 4 \times 0.6 + 9 \times 0.2 = 4.4$
$E[XY] = 1 \times 0.05 + 1 \times 2 \times 0.1 + \cdots + 3 \times 3 \times 0.2 = 4.85$
$cov(X, Y) = E[XY] - E[X]E[Y] = .25$
$var[X] = E[X^2] - E[X]^2 = .51$
$var[Y] = E[Y^2] - E[Y]^2 = .4$
$corr(X, Y) \approx 0.55$

Properties of Covariance

$$cov(X, Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y].$$

Fact
(a) $var[X] = cov(X, X)$
(b) X, Y independent $\Rightarrow cov(X, Y) = 0$
(c) $cov(a + X, b + Y) = cov(X, Y)$
(d) $cov(aX + bY, cU + dV) = ac \cdot cov(X, U) + ad \cdot cov(X, V) + bc \cdot cov(Y, U) + bd \cdot cov(Y, V)$.

Proof:
(a)-(b)-(c) are obvious.
(d) In view of (c), one can subtract the means and assume that the RVs are zero-mean. Then,

$$cov(aX + bY, cU + dV) = E[(aX + bY)(cU + dV)]$$
$$= ac \cdot E[XU] + ad \cdot E[XV] + bc \cdot E[YU] + bd \cdot E[YV]$$
$$= ac \cdot cov(X, U) + ad \cdot cov(X, V) + bc \cdot cov(Y, U) + bd \cdot cov(Y, V).$$

$\square$

Summary

Variance

- Variance: $var[X] := E[(X - E[X])^2] = E[X^2] - E[X]^2$
- Fact: $var[aX + b]a^2 var[X]$
- Sum: X, Y, Z pairwise ind. $\Rightarrow var[X + Y + Z] = \cdots$

Random Variables so far.

Probability Space: Ω, $Pr : \Omega \to [0,1]$, $\sum_{\omega \in \Omega} Pr(w) = 1$.
Random Variables: $X : \Omega \to R$.
Associated event: $Pr[X = a] = \sum_{\omega : X(\omega) = a} Pr(\omega)$
X and Y independent $\iff$ all associated events are independent.
Expectation: $E[X] = \sum_a a Pr[X = a] = \sum_{\omega \in \Omega} X(\omega) Pr(\omega)$.
Linearity: $E[X + Y] = E[X] + E[Y]$.

Variance: $Var(X) = E[(X - E[X])^2] = E[X^2] - (E(X))^2$
For independent X, Y, $Var(X + Y) = Var(X) + Var(Y)$.
Also: $Var(cX) = c^2 Var(X)$ and $Var(X + b) = Var(X)$.

Poisson: $X \sim P(\lambda)$ $E(X) = \lambda$, $Var(X) = \lambda$.
Binomial: $X \sim B(n,p)$ $E(X) = np$, $Var(X) = np(1 - p)$
Uniform: $X \sim U\{1, \ldots, n\}$ $E[X] = \frac{n+1}{2}$, $Var(X) = \frac{n^2-1}{12}$.
Geometric: $X \sim G(p)$ $E(X) = \frac{1}{p}$, $Var(X) = \frac{1-p}{p^2}$