Outline

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Regression (optional.)

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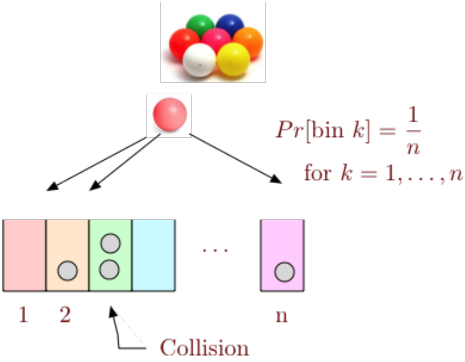

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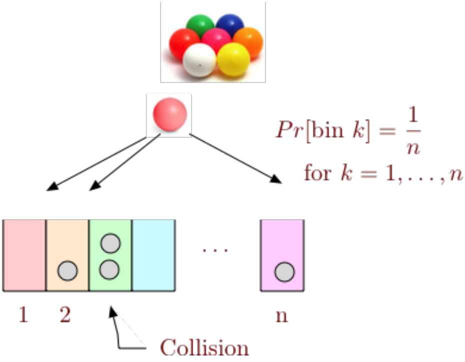

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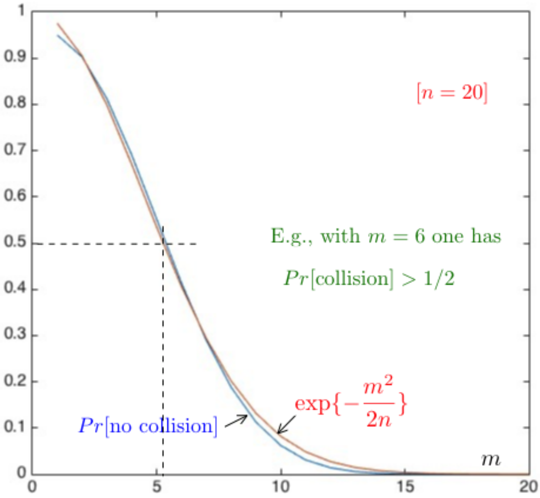

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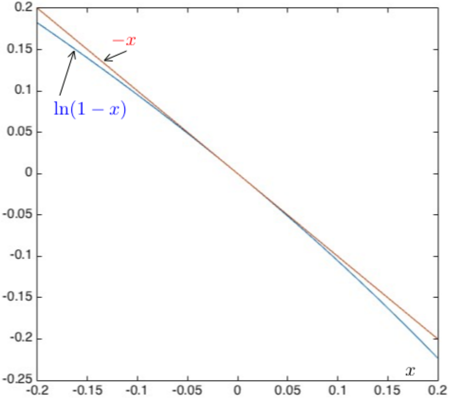
(†) $1 + 2 + \cdots + m - 1 = (m-1)m/2$.

Approximation



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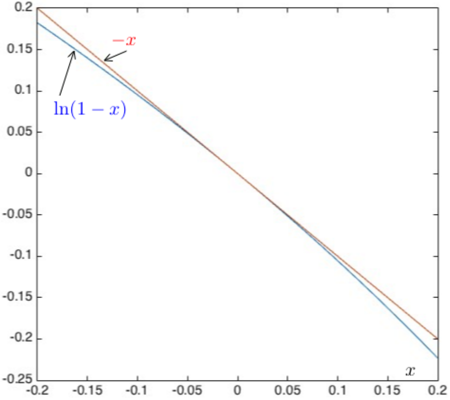

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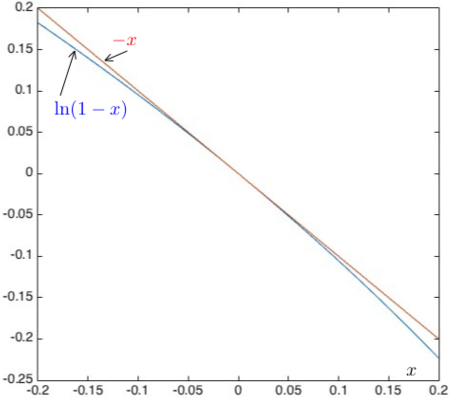

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Hence, $-x \approx \ln(1-x)$ for $|x| \ll 1$.

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Pr[\text{no collision}] \approx 1 - 10^{-3} &\Leftrightarrow m^2/(2n) \approx 10^{-3} \\
&\Leftrightarrow 2n \approx m^2 10^3 \Leftrightarrow 2^{b+1} \approx m^2 2^{10} \\
&\Leftrightarrow b + 1 \approx 10 + 2\log_2(m) \approx 10 + 2.9 \ln(m).
\end{aligned}
$$

Note: $\log_2(x) = \log_2(e) \ln(x) \approx 1.44 \ln(x)$.

Coupon Collector Problem.

There are n different baseball cards.
(Brian Wilson, Jackie Robinson, Roger Hornsby, ...)

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Theorem: If you buy m boxes,

(a) $Pr[\text{miss one specific item}] \approx e^{-\frac{m}{n}}$

(b) $Pr[\text{miss any one of the items}] \leq ne^{-\frac{m}{n}}$.

Coupon Collector Problem: Analysis.

Event A_m = 'fail to get Brian Wilson in m cereal boxes'

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Pr[A_m] &= (1-\frac{1}{n}) \times \cdots \times (1-\frac{1}{n}) \\
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For $p_m = \frac{1}{2}$, we need around $n \ln 2 \approx 0.69n$ boxes.

Collect all cards?

Experiment: Choose m cards at random with replacement.

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Probability of failing to get at least one of these *n* players:

$$p := Pr[E_1 \cup E_2 \cdots \cup E_n]$$

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Plug in and get

$$p \leq ne^{-\frac{m}{n}}.$$

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E.g., $n = 10^2 \Rightarrow m = 530$;

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E.g., $n = 10^2 \Rightarrow m = 530; n = 10^3 \Rightarrow m = 7600$.

Time to collect coupons

X-time to get n coupons.

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X_1 - time to get first coupon.

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Pr[“get second coupon”|“got milk ”]

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Pr[“get second coupon”|“got ~~milk~~ first coupon”] $= \frac{n-1}{n}$

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Review: Harmonic sum

$$H(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \int_1^n \frac{1}{x} dx = \ln(n).$$

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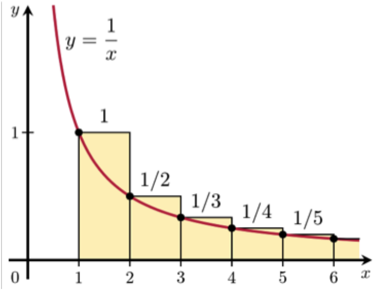

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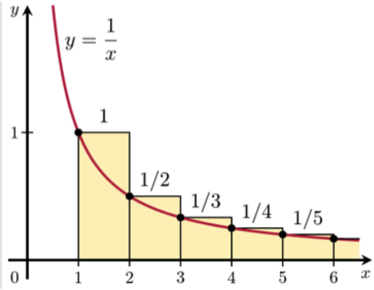

A good approximation is

$$H(n) \approx \ln(n) + \gamma \text{ where } \gamma \approx 0.58 \text{ (Euler-Mascheroni constant).}$$

Simplest..

Load balance: m balls in n bins.

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Round robin:

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Max load with probability $\geq 1 - \delta$?

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$\delta = \frac{1}{n^c}$ for today.

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Better than variance based methods...

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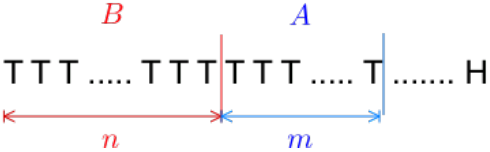

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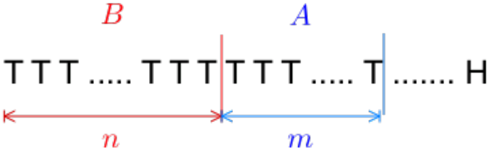

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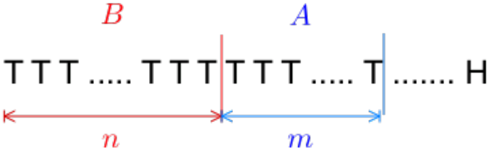

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The coin is memoryless, therefore, so is X.

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For $X = P(\lambda)$ and $Y = P(\mu)$, what is $X + Y$?

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Details: both could arrive with probability $\lambda\mu/n^2$.
But this goes to zero as $n \to \infty$.
(Like λ^2/n^2 in previous derivation)

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A bit later, we will consider a general function $g(X)$.

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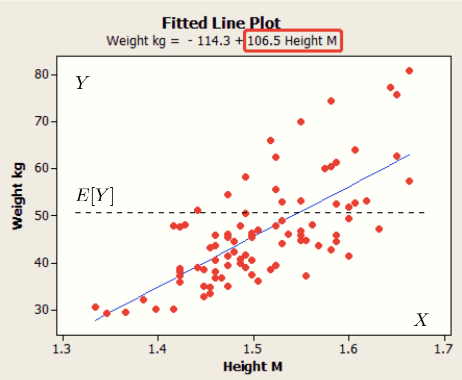

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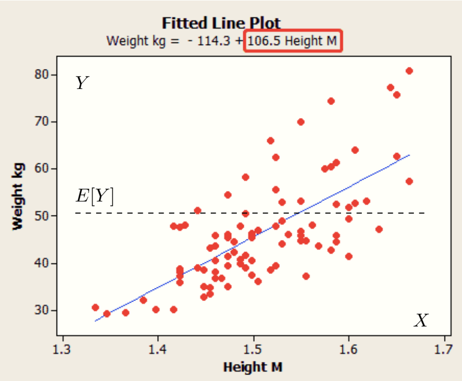

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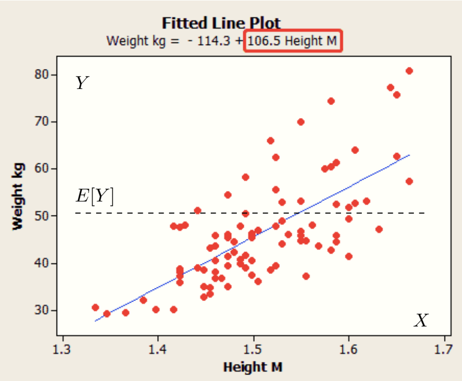

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Best linear fit: Linear Regression.

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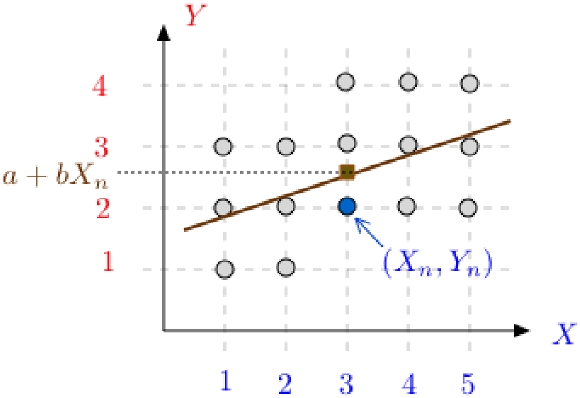

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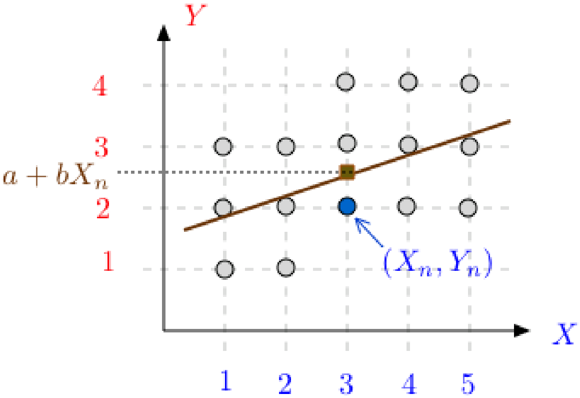

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$^{(*)}$ Recall that $cov(X,Y) = E[(X - E[X])(Y - E[Y])]$ and $var[X] = E[(X - E[X])^2]$.

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Probability Space: Ω, $Pr : \Omega \rightarrow [0,1]$, $\sum_{\omega \in \Omega} Pr(w) = 1$.

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Inference:
Have one of two coins. Flip coin, which coin do you have?
Got positive test result. What is probability you have disease?

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Uniform: $X \sim U\{1,\ldots,n\}$ $\quad \forall i \in [1,n], Pr[X = i] = \frac{1}{n}$.

$E[X] = \frac{n+1}{2}$, $Var(X) = \frac{n^2-1}{12}$.

Geometric: $X \sim G(p)$ $\quad Pr[X = i] = (1-p)^{i-1}p$

Random Variables

Random Variables: $X : \Omega \to R$.

Distribution: $Pr[X = a] = \sum_{\omega : X(\omega)=a} Pr(\omega)$

X and Y independent $\iff$ all associated events are independent.
Expectation: $E[X] = \sum_a aPr[X = a] = \sum_{\omega \in \Omega} X(\omega)Pr(\omega)$.
Linearity: $E[X + Y] = E[X] + E[Y]$.

Variance: $Var(X) = E[(X - E[X])^2] = E[X^2] - (E(X))^2$
For independent X, Y, $Var(X + Y) = Var(X) + Var(Y)$.
Also: $Var(cX) = c^2 Var(X)$ and $Var(X + b) = Var(X)$.

Poisson: $X \sim P(\lambda)$ $\quad Pr[X = i] = e^{-\lambda} \frac{\lambda^i}{i!}$.
$E(X) = \lambda$, $Var(X) = \lambda$.
Binomial: $X \sim B(n, p)$ $\quad Pr[X = i] = \binom{n}{i} p^i (1 - p)^{n-i}$
$E(X) = np$, $Var(X) = np(1 - p)$
Uniform: $X \sim U\{1, \ldots, n\}$ $\quad \forall i \in [1, n], Pr[X = i] = \frac{1}{n}$.
$E[X] = \frac{n+1}{2}$, $Var(X) = \frac{n^2-1}{12}$.
Geometric: $X \sim G(p)$ $\quad Pr[X = i] = (1 - p)^{i-1} p$
$E(X) = \frac{1}{p}$, $Var(X) = \frac{1-p}{p^2}$

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Note: Probability Mass Function $\equiv$ Distribution.