CS70: Continuous Probability.

Continuous Probability 1

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1. Examples
2. Events
3. Continuous Random Variables

Uniformly at Random in $[0, 1]$.

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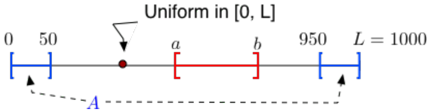

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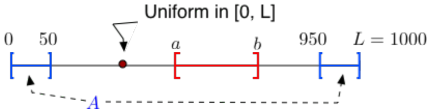

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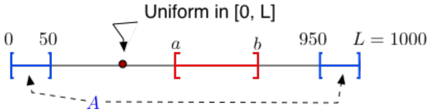

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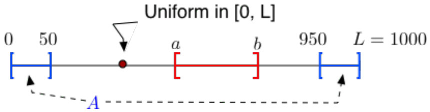

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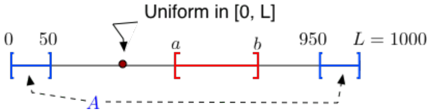

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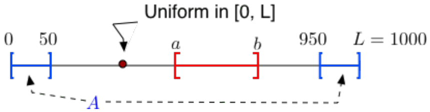

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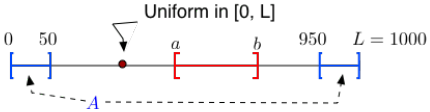

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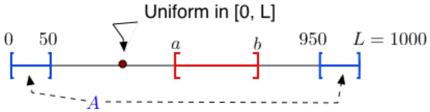

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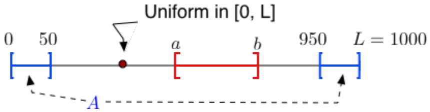

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Makes sense: $b - a$ is the fraction of $[0,1]$ that $[a,b]$ covers.

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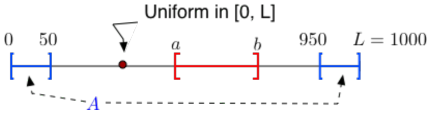

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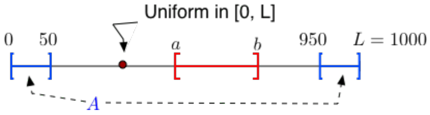

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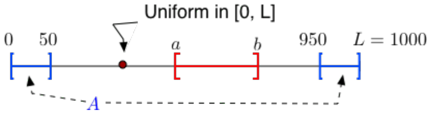

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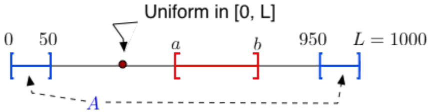

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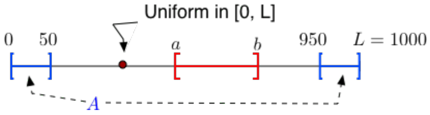

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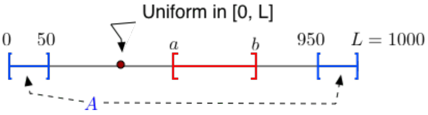

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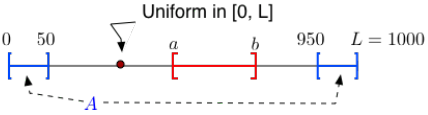

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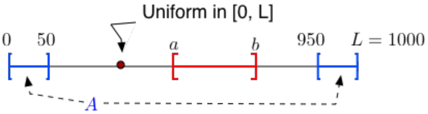

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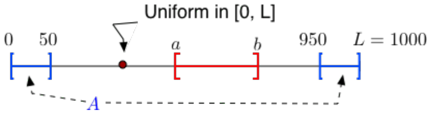

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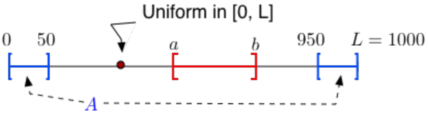

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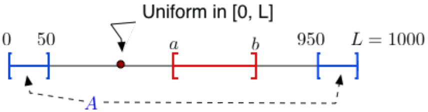

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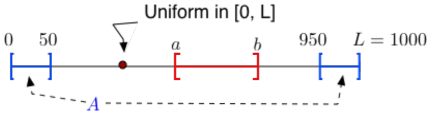

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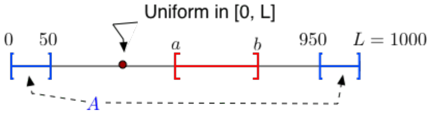

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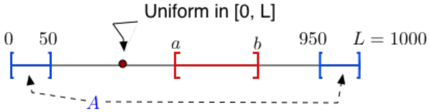

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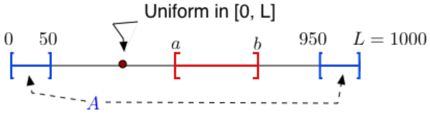

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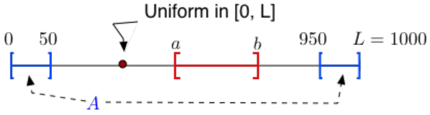

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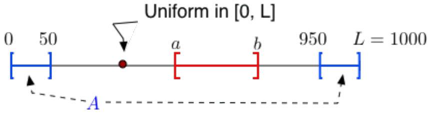

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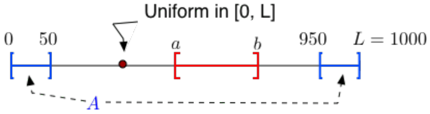

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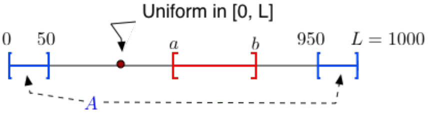

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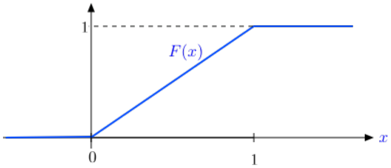

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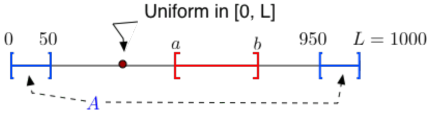

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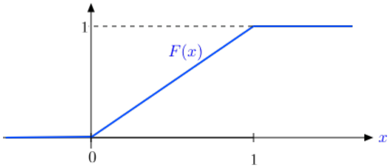
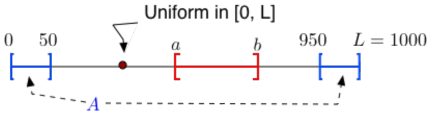


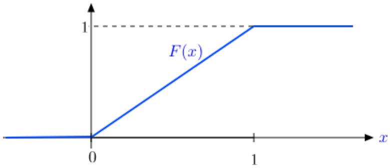
Then we have $Pr[X \in (a,b]] = Pr[X \leq b] - Pr[X \leq a]$

Uniformly at Random in [0, 1].



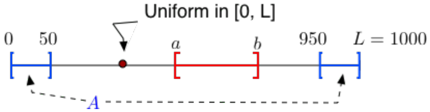
$Pr[X \leq x] = x$ for $x \in [0,1]$. Also, $Pr[X \leq x] = 0$ for $x < 0$.
$Pr[X \leq x] = 1$ for $.2x > 1$.

Define $F(x) = Pr[X \leq x]$.



Then we have $Pr[X \in (a,b]] = Pr[X \leq b] - Pr[X \leq a] = F(b) - F(a)$.

Uniformly at Random in $[0,1]$.



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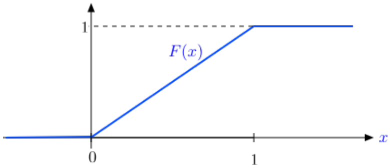
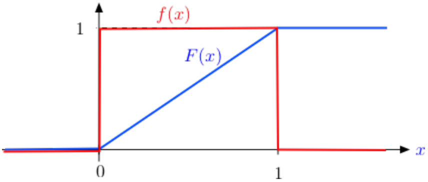


Then we have $Pr[X \in (a,b]] = Pr[X \leq b] - Pr[X \leq a] = F(b) - F(a)$.

Thus, $F(\cdot)$ specifies the probability of all the events!

Uniformly at Random in $[0,1]$.



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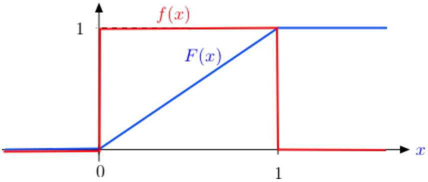

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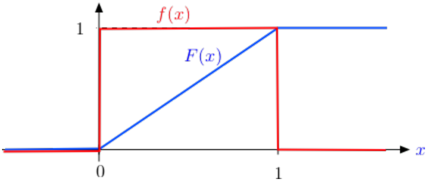
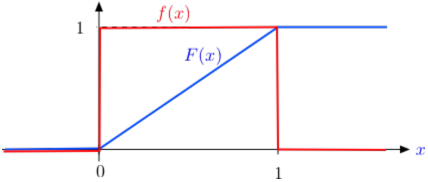


$$Pr[X \in (a,b]] = Pr[X \leq b] - Pr[X \leq a] = F(b) - F(a).$$

An alternative view is to define $f(x) = \frac{d}{dx}F(x) =$

Uniformly at Random in $[0,1]$.



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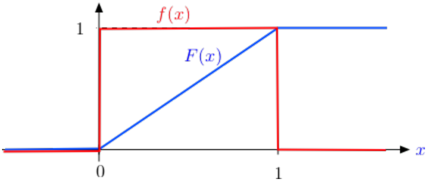

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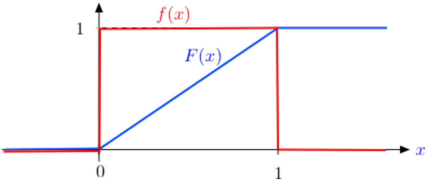

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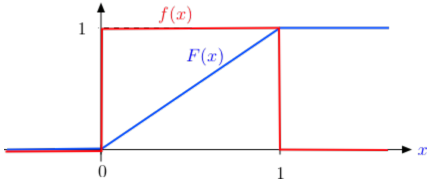
Thus, the probability of an event is the integral of $f(x)$ over the event:

Uniformly at Random in $[0, 1]$.



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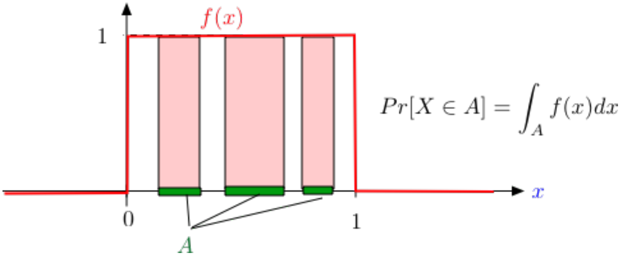
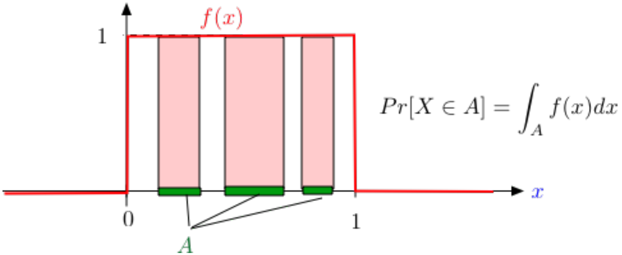
Thus, the probability of an event is the integral of $f(x)$ over the event:

$$Pr[X \in A] = \int_A f(x)dx.$$

Uniformly at Random in $[0, 1]$.

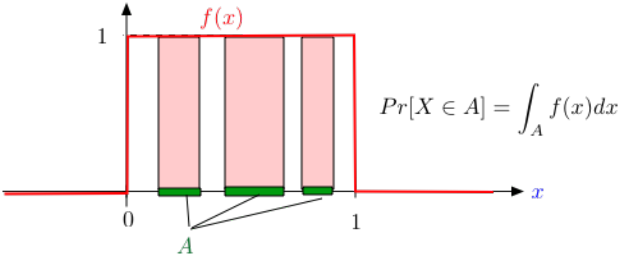


Uniformly at Random in $[0, 1]$.



Think of $f(x)$ as describing how
one unit of probability is spread over $[0, 1]$:

Uniformly at Random in $[0, 1]$.



Think of $f(x)$ as describing how
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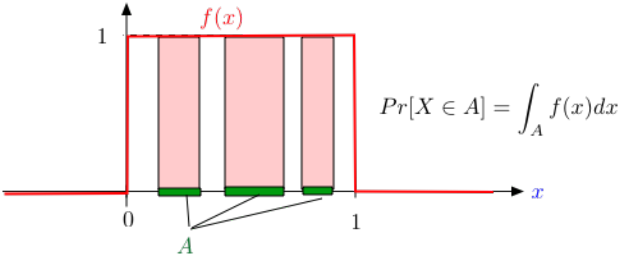
Uniformly at Random in $[0,1]$.



Think of $f(x)$ as describing how
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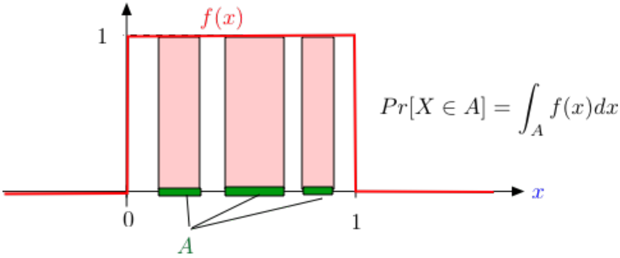
Then $Pr[X \in A]$ is the probability mass over A.

Uniformly at Random in $[0, 1]$.



Think of $f(x)$ as describing how
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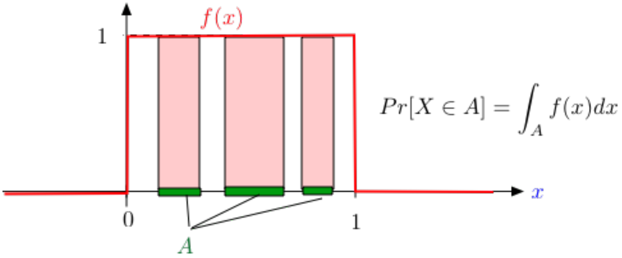
Then $Pr[X \in A]$ is the probability mass over A.

Observe:

Uniformly at Random in $[0, 1]$.



Think of $f(x)$ as describing how
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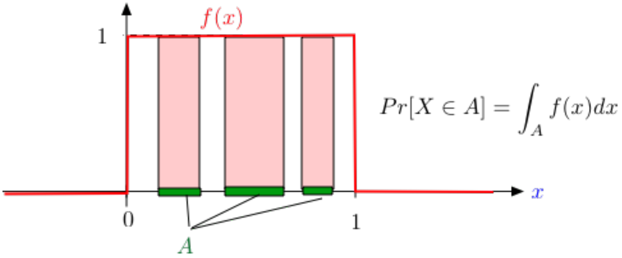
Observe:

- This makes the probability automatically additive.

Uniformly at Random in $[0, 1]$.



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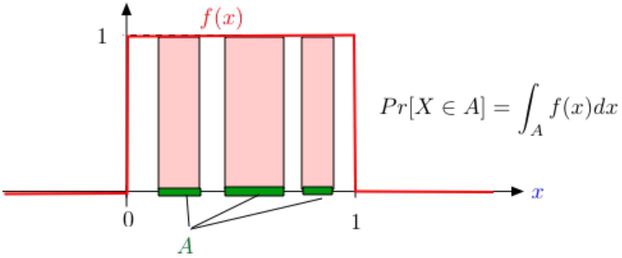
- We need $f(x) \geq 0$

Uniformly at Random in $[0, 1]$.



Think of $f(x)$ as describing how
one unit of probability is spread over $[0, 1]$: uniformly!

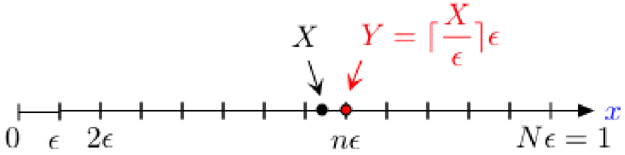
Then $Pr[X \in A]$ is the probability mass over A.

Observe:

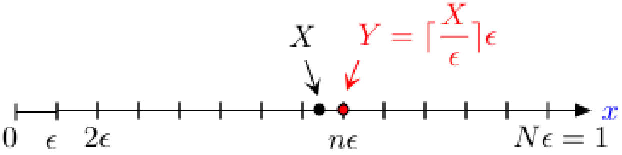
- This makes the probability automatically additive.
- We need $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$.

Uniformly at Random in $[0, 1]$.

Uniformly at Random in $[0, 1]$.

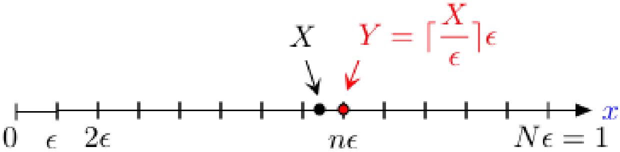


Uniformly at Random in $[0, 1]$.



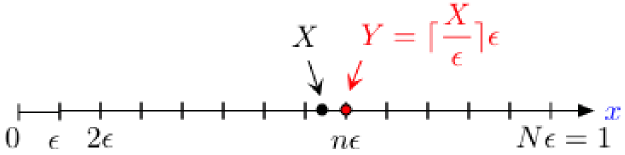
Discrete Approximation:

Uniformly at Random in $[0, 1]$.



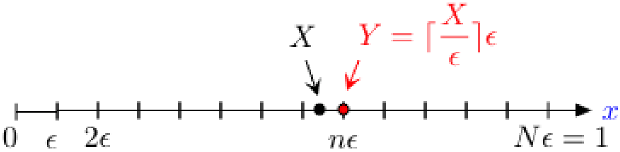
Discrete Approximation: Fix $N \gg 1$

Uniformly at Random in $[0, 1]$.



Discrete Approximation: Fix $N \gg 1$ and let $\varepsilon = 1/N$.

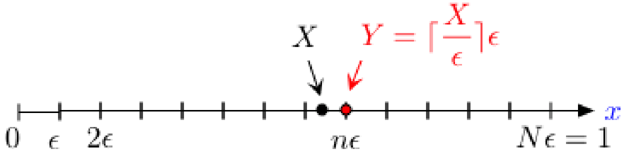
Uniformly at Random in $[0,1]$.



Discrete Approximation: Fix $N \gg 1$ and let $\varepsilon = 1/N$.

Define $Y = n\varepsilon$ if $(n-1)\varepsilon < X \leq n\varepsilon$ for $n = 1, \ldots, N$.

Uniformly at Random in $[0,1]$.

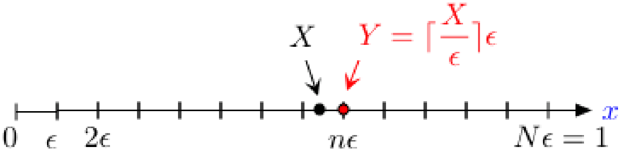


Discrete Approximation: Fix $N \gg 1$ and let $\varepsilon = 1/N$.

Define $Y = n\varepsilon$ if $(n-1)\varepsilon < X \leq n\varepsilon$ for $n = 1, \ldots, N$.

Then $|X - Y| \leq \varepsilon$

Uniformly at Random in $[0, 1]$.

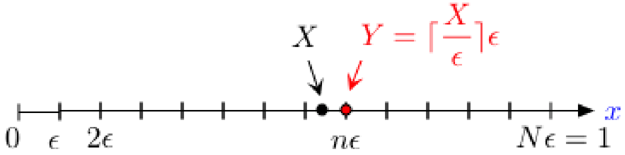


Discrete Approximation: Fix $N \gg 1$ and let $\varepsilon = 1/N$.

Define $Y = n\varepsilon$ if $(n-1)\varepsilon < X \leq n\varepsilon$ for $n = 1, \ldots, N$.

Then $|X - Y| \leq \varepsilon$ and Y is discrete:

Uniformly at Random in $[0,1]$.



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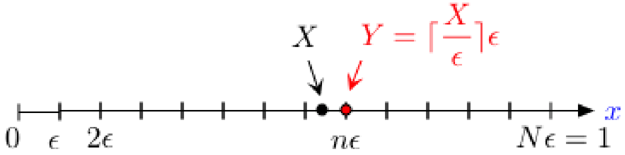
Then $|X - Y| \leq \varepsilon$ and Y is discrete: $Y \in \{\varepsilon, 2\varepsilon, \ldots, N\varepsilon\}$.

Uniformly at Random in $[0,1]$.



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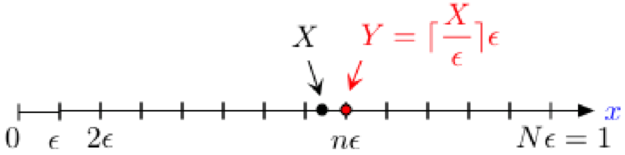
Then $|X - Y| \le \varepsilon$ and Y is discrete: $Y \in \{\varepsilon, 2\varepsilon, \ldots, N\varepsilon\}$.

Also, $Pr[Y = n\varepsilon] = \frac{1}{N}$ for $n = 1, \ldots, N$.

Uniformly at Random in $[0,1]$.



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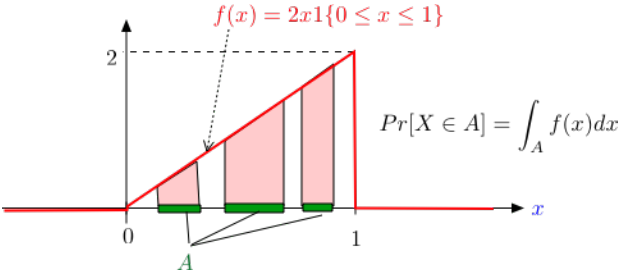
Then $|X - Y| \leq \varepsilon$ and Y is discrete: $Y \in \{\varepsilon, 2\varepsilon, \ldots, N\varepsilon\}$.

Also, $Pr[Y = n\varepsilon] = \frac{1}{N}$ for $n = 1, \ldots, N$.

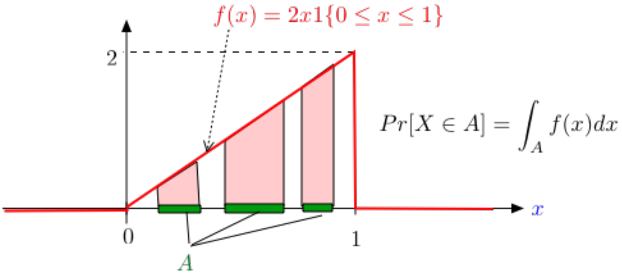
Thus, X is ‘almost discrete.’

Nonuniformly at Random in $[0, 1]$.

Nonuniformly at Random in $[0, 1]$.

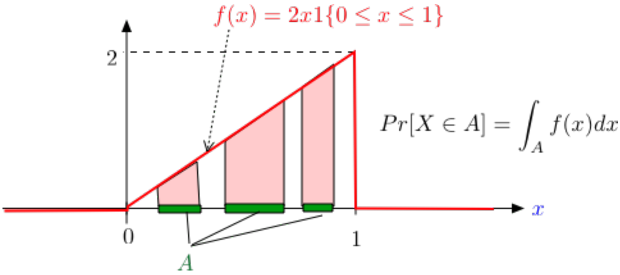


Nonuniformly at Random in $[0, 1]$.



This figure shows a different choice of $f(x) \geq 0$ with $\int_{-\infty}^{\infty} f(x)dx = 1$.

Nonuniformly at Random in $[0, 1]$.



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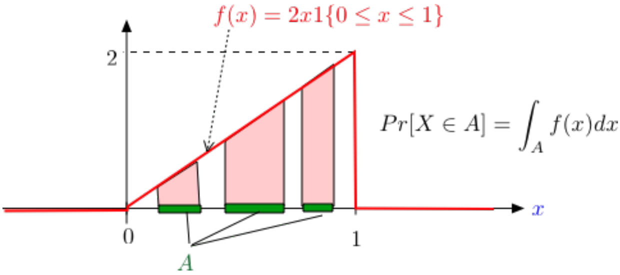
It defines another way of choosing X at random in $[0, 1]$.

Nonuniformly at Random in $[0, 1]$.



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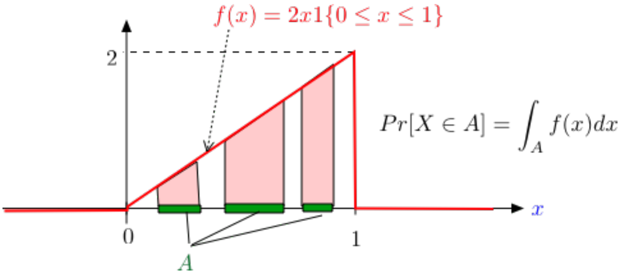
Note that X is more likely to be closer to 1 than to 0.

Nonuniformly at Random in $[0, 1]$.



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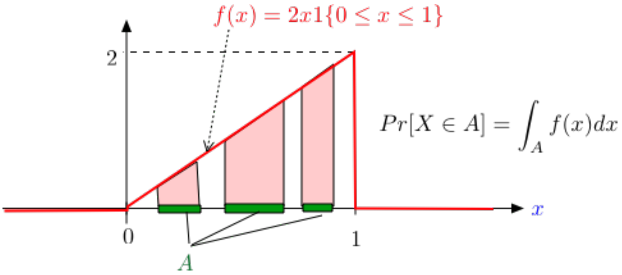

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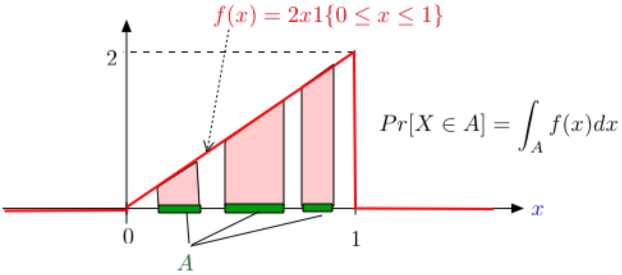
One has $Pr[X \leq x] = \int_{-\infty}^{x} f(u)du = x^2$

Nonuniformly at Random in $[0, 1]$.



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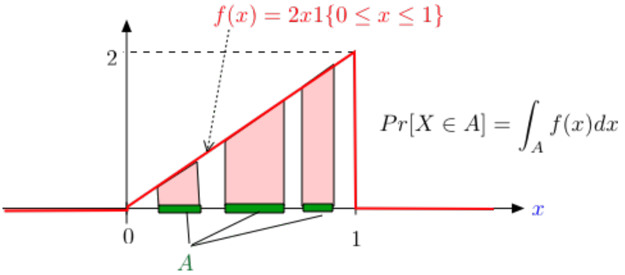
One has $Pr[X \leq x] = \int_{-\infty}^{x} f(u)du = x^2$ for $x \in [0, 1]$.

Nonuniformly at Random in $[0,1]$.



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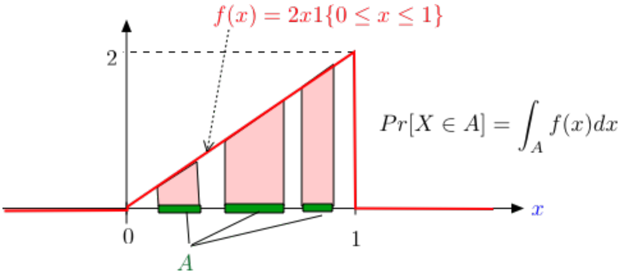
One has $Pr[X \le x] = \int_{-\infty}^{x} f(u)du = x^2$ for $x \in [0,1]$.

Also, $Pr[X \in (x, x+\varepsilon)] = \int_{x}^{x+\varepsilon} f(u)du$

Nonuniformly at Random in $[0, 1]$.



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Also, $Pr[X \in (x, x + \varepsilon)] = \int_{x}^{x+\varepsilon} f(u)du \approx f(x)\varepsilon$.

Another Nonuniform Choice at Random in $[0, 1]$.

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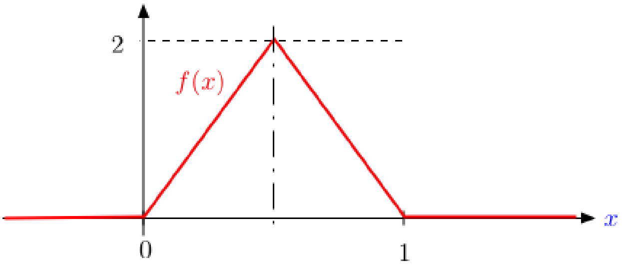

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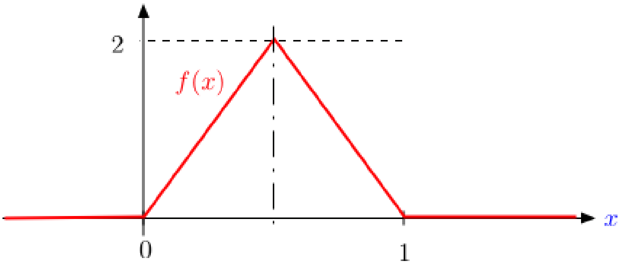

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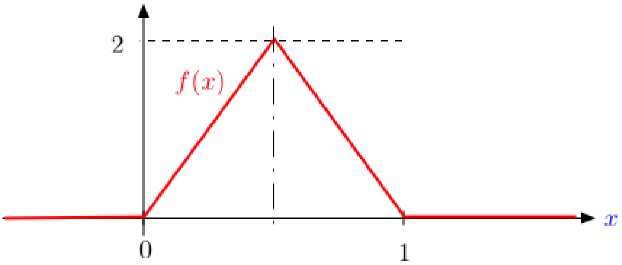

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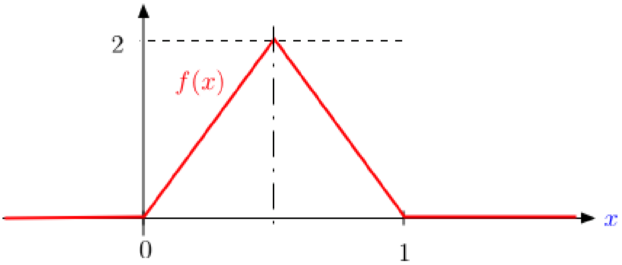

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Note that X is more likely to be closer to $1/2$ than to 0 or 1.

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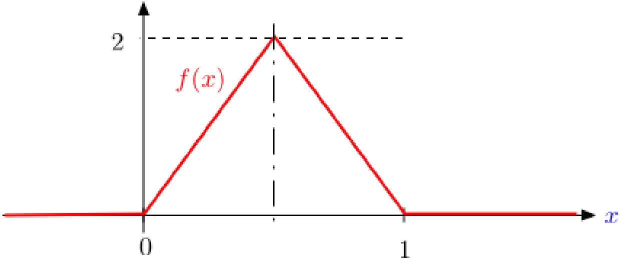

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For instance, $Pr[X \in [0,1/3]] =$

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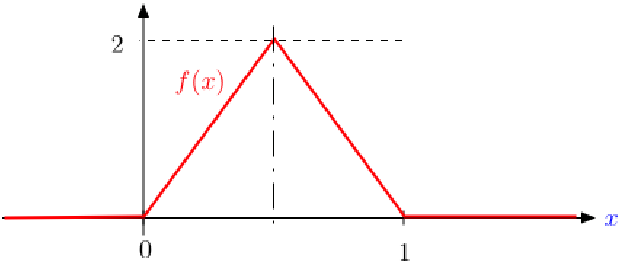

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Note that X is more likely to be closer to $1/2$ than to 0 or 1.

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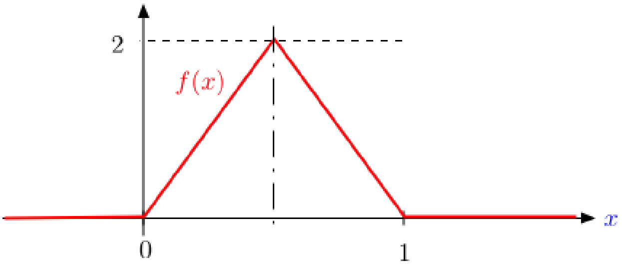
Another Nonuniform Choice at Random in $[0, 1]$.



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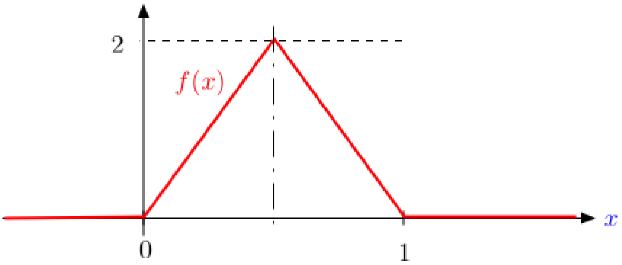

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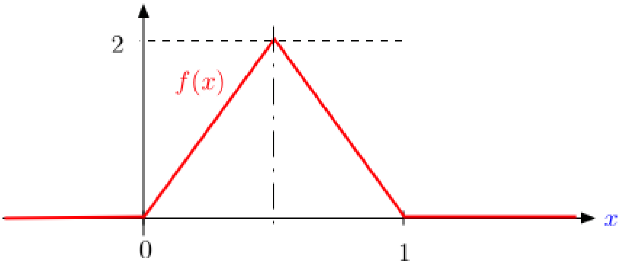

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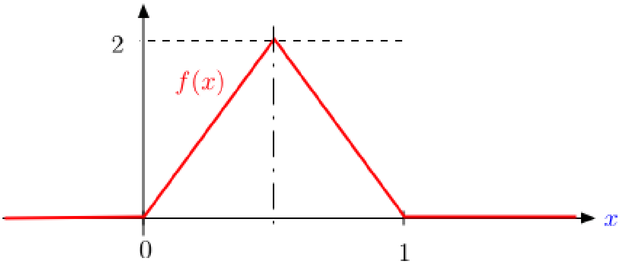

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General Random Choice in $\mathfrak{R}$

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To indicate that F and f correspond to the RV X, we will write them $F_X(x)$ and $f_X(x)$.

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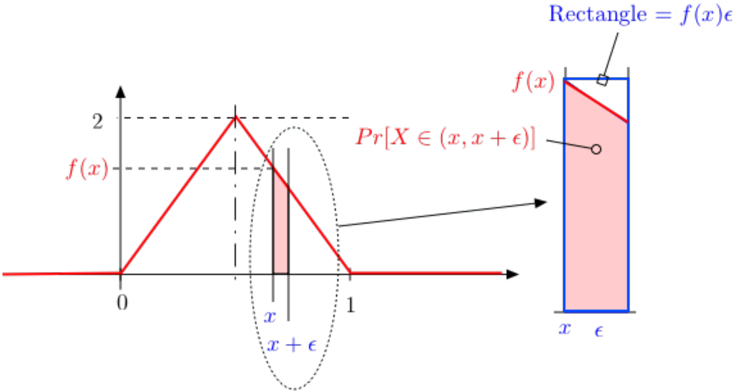

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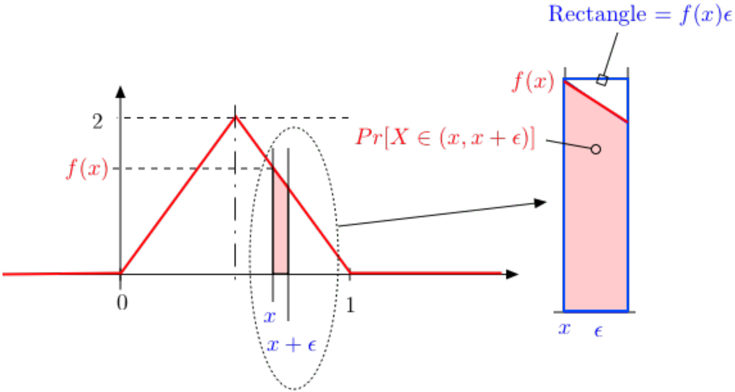

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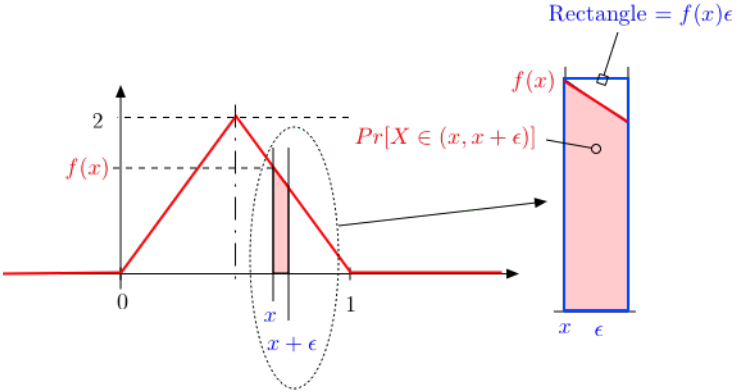

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Example: hitting random location on gas tank.

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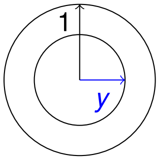

Random Variable: Y distance from center.

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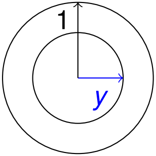

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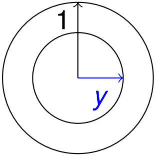

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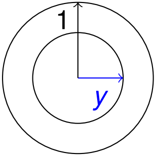

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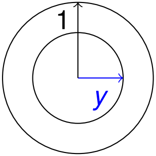

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Calculation of event with dartboard..

Probability between .5 and .6 of center?

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PDF.

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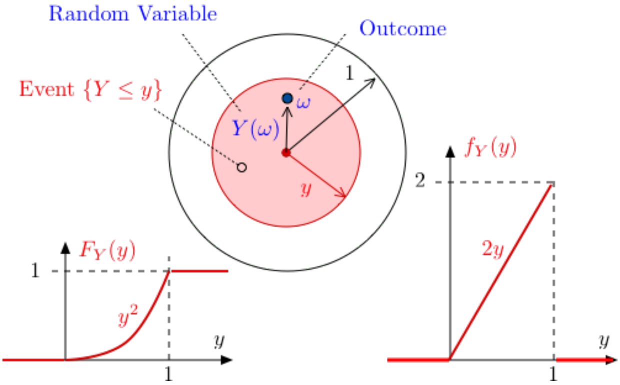
The cumulative distribution function (cdf) and probability distribution function (pdf) give full information.
Use whichever is convenient.

Target

Target



Random Variable
Outcome
Event $\{Y \leq y\}$
ω
$Y(\omega)$
1
y
$f_Y(y)$
2
$2y$
y
1
$F_Y(y)$
1
y^2
y
1

$U[a, b]$

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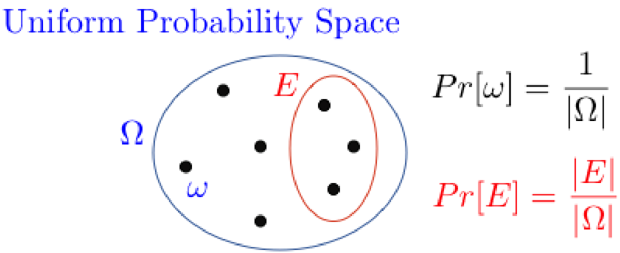
Uniform Probability Space
Ω
ω
E
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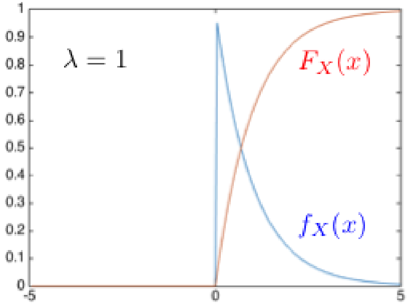

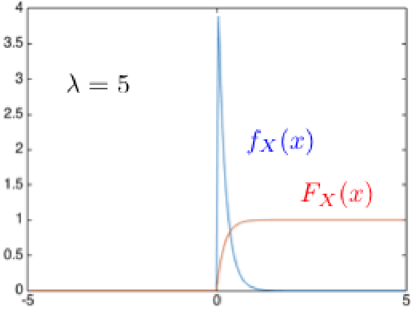

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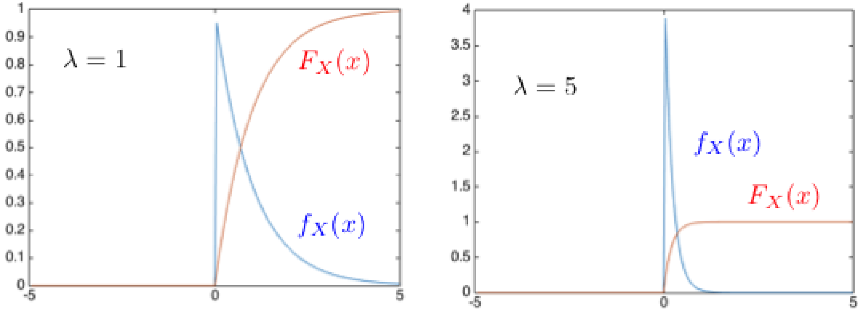

Note that $Pr[X > t] = e^{-\lambda t}$ for $t > 0$.

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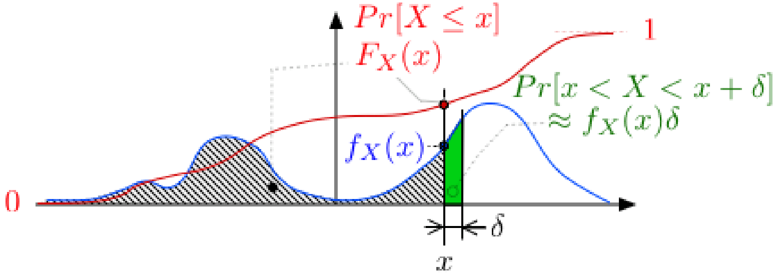
X "takes" value $n\delta$, for $n \in Z$, with $Pr[X = n\delta] = f_X(n\delta)\delta$

A Picture



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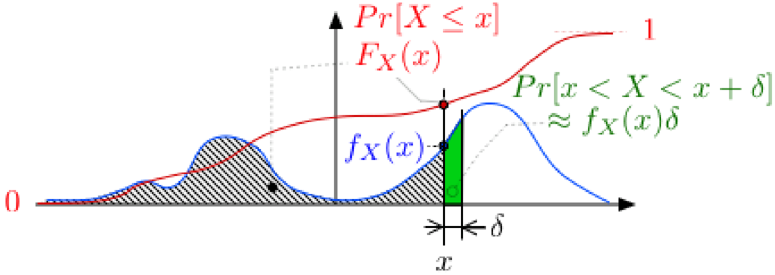

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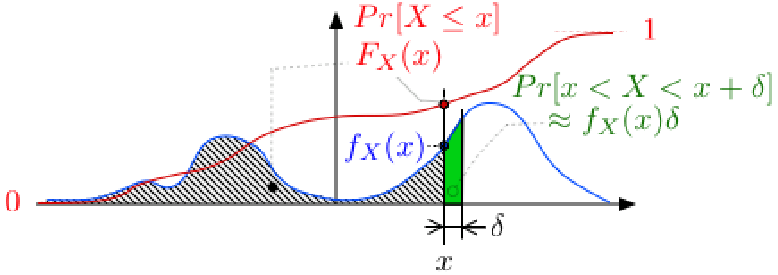

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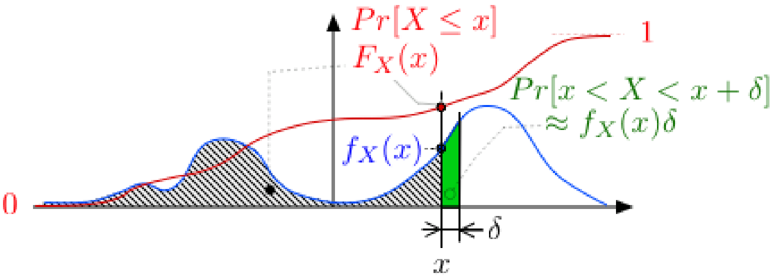

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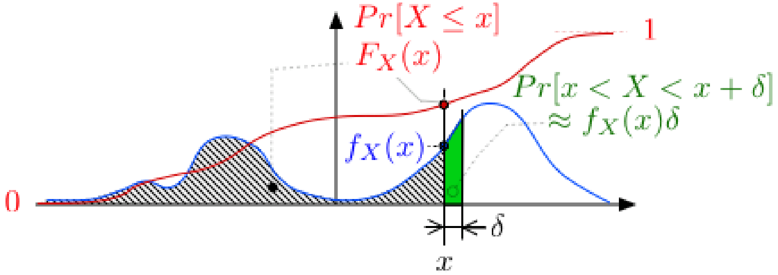

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Extension: $\mathbf{X} = (X_1, \ldots, X_n)$ with $f_{\mathbf{X}}(\mathbf{x})$.

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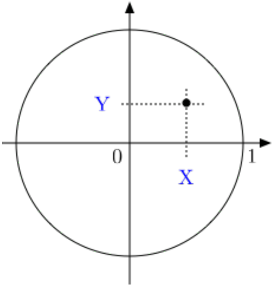

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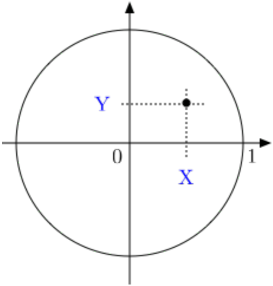

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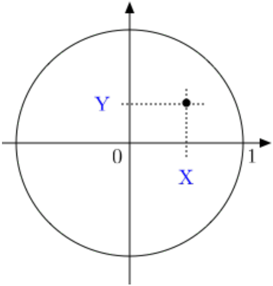

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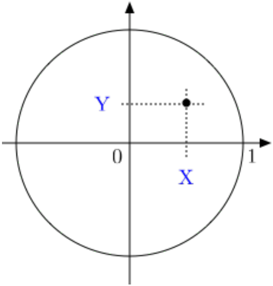

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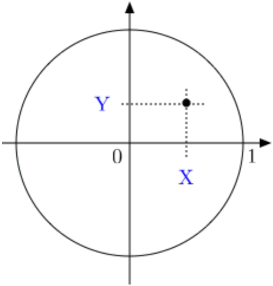

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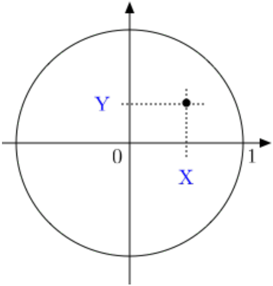

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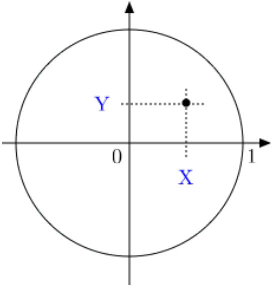

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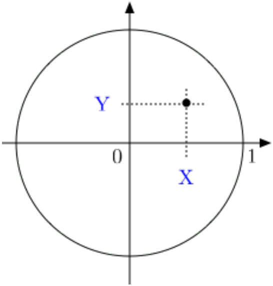

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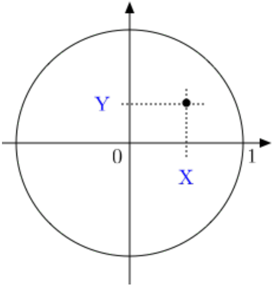

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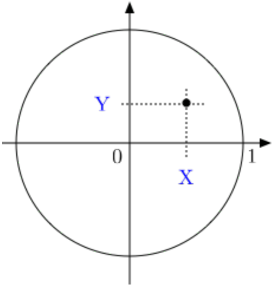

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$$Pr[X^2 + Y^2 \leq r^2] = \frac{\pi r^2}{\pi} = r^2$$

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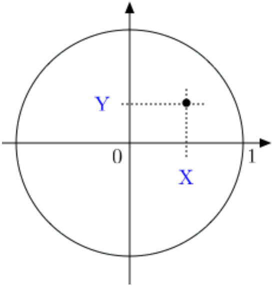
Example of Continuous (X, Y)

Pick a point (X, Y) uniformly in the unit circle.



Thus, $f_{X,Y}(x,y) = \frac{1}{\pi} 1\{x^2 + y^2 \leq 1\}$.

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Independent Continuous Random Variables

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Thus, $f_{X,Y}(x,y) = f_X(x) f_Y(y)$.

□

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Proof: As in the discrete case.

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Corollary: For independent random variables, $f_{X|Y}(x,y) = f_X(x)$.

Independent Random Variables?

Uniform on a rectangle?

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Uniform on a rectangle? Independent?

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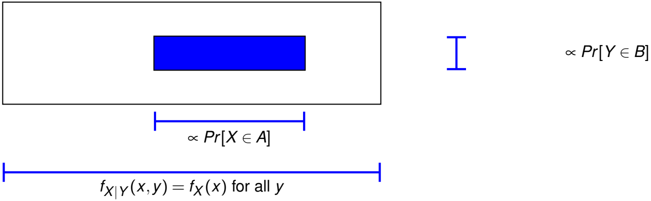

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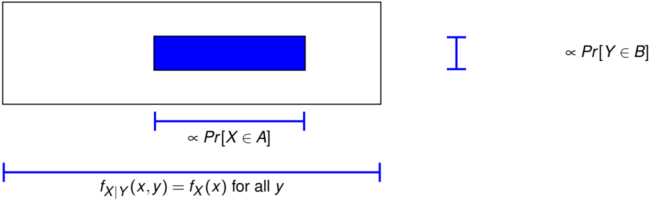

Also: $Pr[X \in A, Y \in B] \propto$ Area of rectangle $\propto Pr[X \in A] \times Pr[Y \in B]$.

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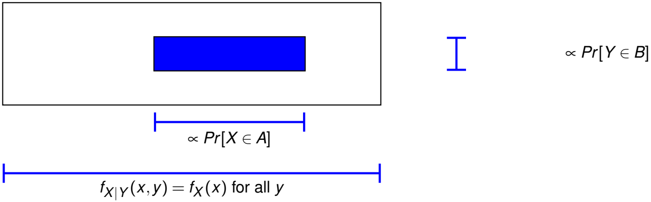

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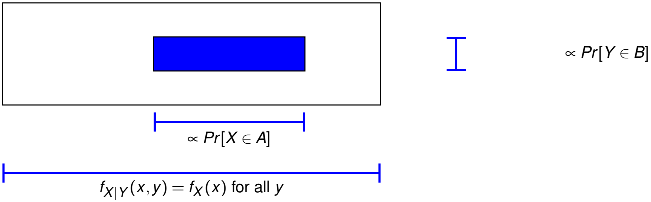

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Uniform on a circle?

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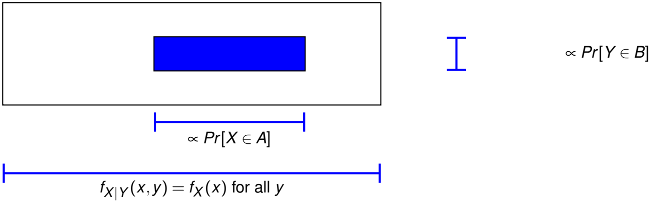
Uniform on a rectangle? Independent?



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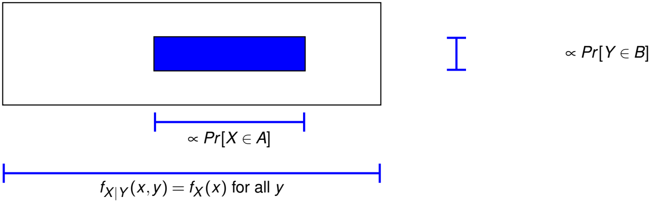

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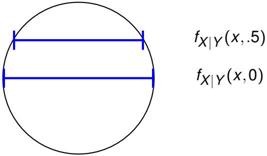

Not independent!

Summary

Continuous Probability 1

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6. Joint pdf: $Pr[X \in (x, x+\delta), Y = (y, y+\delta)) = f_{X,Y}(x,y)\delta^2$.
   - 6.1 Conditional Distribution: $f_{X|Y}(x,y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$.
   - 6.2 Independence: $f_{X|Y}(x,y) = f_X(x)$