Lecture Today.

To homework (score) or not to homework (score)

Do proofs of optimality/pessimality again.

Graphs

Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

Theorem: Job Propose and Reject produces a job-optimal pairing.

Proof:
Assume not: there is a job b does not get optimal candidate, g.

There is a stable pairing S where b and g are paired.

Let t be first day job b gets rejected
by its optimal candidate g who it is paired with
in stable pairing S.

b^* - knocks b off of g's string on day t $\implies$ g prefers b^* to b

By choice of t, b^* likes g at least as much as optimal candidate.

$\implies$ b^* prefers g to its partner g^* in S.

Rogue couple for S.
So S is not a stable pairing. Contradiction. □

Notes: S - stable. $(b^*, g^*) \in S$. But (b^*, g) is rogue couple!

Used Well-Ordering principle...Induction.

How about for candidates?

Theorem: Job Propose and Reject produces candidate-pessimal pairing.

T – pairing produced by JPR.

S – worse stable pairing for candidate g.

In T, (g,b) is pair.

In S, (g,b^*) is pair.

g prefers b to b^*.

T is job optimal, so b prefers g to its partner in S.

(g,b) is Rogue couple for S

S is not stable.

Contradiction. □

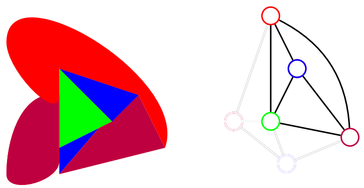
Notes: Not really induction.
Structural statement: Job optimality $\implies$ Candidate pessimality.

Lecture 5: Graphs.

Graphs!
- Definitions: model.
- Fact!

Planar graphs.
- Euler Again!!!!

Map Coloring.



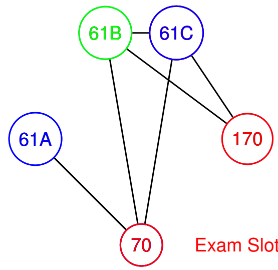
Four colors required!

Theorem: Four colors enough.

Fewer Colors?

Yes! Three colors.

Scheduling: coloring.

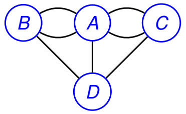


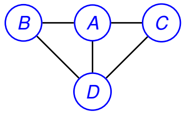
Exam Slot 1.

Exam Slot 2.

Exam Slot 3.

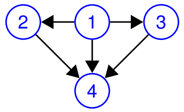
Graphs: formally.





Graph: $G = (V, E)$.
V - set of vertices.
$\{A, B, C, D\}$
$E \subseteq V \times V$ - set of edges.
$\{\{A,B\}, \{A,B\}, \{A,C\}, \{A,C\}, \{B,D\}, \{A,D\}, \{C,D\}\}$.
For CS 70, usually simple graphs.
No parallel edges.
Multigraph above.

Directed Graphs



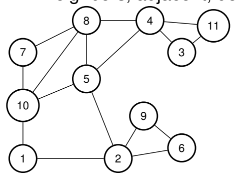
$G = (V, E)$.
V - set of vertices.
$\{1, 2, 3, 4\}$
E ordered pairs of vertices.
$\{(1,2), (1,3), (1,4), (2,4), (3,4)\}$

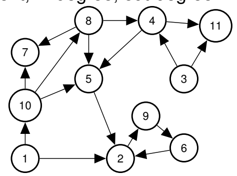
One way streets.
Tournament: 1 beats 2, ...
Precedence: 1 is before 2, ..

Social Network: Directed? Undirected?
Friends. Undirected.
Likes. Directed.

Graph Concepts and Definitions.

Graph: $G = (V, E)$
neighbors, adjacent, degree, incident, in-degree, out-degree





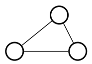
Neighbors of 10? 1,5,7, 8.
u is neighbor of v if $\{u, v\} \in E$.
Edge $\{10, 5\}$ is incident to vertex 10 and vertex 5.
Edge $\{u, v\}$ is incident to u and v.
Degree of vertex 1? 2
Degree of vertex u is number of incident edges.
Equals number of neighbors in simple graph.

Directed graph?
In-degree of 10? 1 Out-degree of 10? 3

Sum of degrees?

The sum of the vertex degrees is equal to

(A) the total number of vertices, $|V|$.
(B) the total number of edges, $|E|$.
(C) What?



Not (A)! Triangle.
Not (B)! Triangle.

What? For triangle number of edges is 3, the sum of degrees is 6.

Could it always be...$2|E|$? ..or $2|V|$?

Quick Proof of an Equality.

The sum of the vertex degrees is equal to ??

Recall:

edge, (u, v), is incident to endpoints, u and v.

degree of u number of edges incident to u

Let's count incidences in two ways.

How many incidences does each edge contribute? 2.

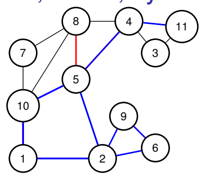
Total Incidences? $|E|$ edges, 2 each. $\rightarrow 2|E|$

What is degree v? Incidences corresponding to v!

Total Incidences? The sum over vertices of degrees!

Thm: Sum of vertex degress is $2|E|$.

Paths, walks, cycles, tour.

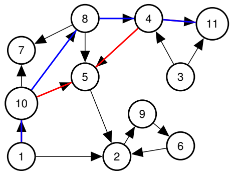


A path in a graph is a sequence of edges.

Path? $\{1, 10\}$, $\{8, 5\}$, $\{4, 5\}$? No!
Path? $\{1, 10\}$, $\{10, 5\}$, $\{5, 4\}$, $\{4, 11\}$? Yes!
Path: $(v_1, v_2), (v_2, v_3), \ldots (v_{k-1}, v_k)$.
Quick Check! Length of path? k vertices or $k-1$ edges.
Cycle: Path with $v_1 = v_k$. Length of cycle? $k-1$ vertices and edges!
Path is usually simple. No repeated vertex!
Walk is sequence of edges with possible repeated vertex or edge.
Tour is walk that starts and ends at the same node.

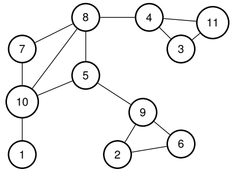
Quick Check!
Path is to Walk as Cycle is to ?? Tour!

Directed Paths.



Path: $(v_1, v_2), (v_2, v_3), \ldots (v_{k-1}, v_k)$.
Paths, walks, cycles, tours ... are analagous to undirected now.

Connectivity: undirected graph.



u and v are connected if there is a path between u and v.

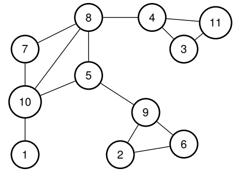
A connected graph is a graph where all pairs of vertices are connected.

If one vertex x is connected to every other vertex.
Is graph connected? Yes? No?

Proof: Use path from u to x and then from x to v. □

May not be simple!
Either modify definition to walk.
Or cut out cycles. .

Connected Components: Quiz.



Is graph above connected? Yes!

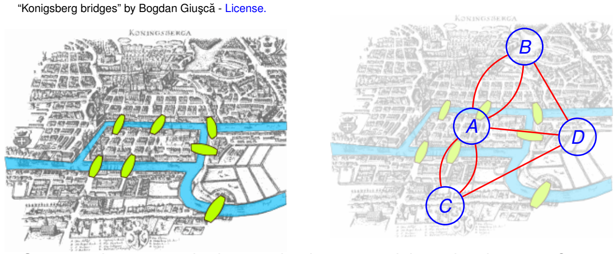
How about now? No!

Connected Components? $\{1\}, \{10, 7, 5, 8, 4, 3, 11\}, \{2, 9, 6\}$.
Connected component - maximal set of connected vertices.
Quick Check: Is $\{10, 7, 5\}$ a connected component? No.

Konigsberg bridges problem.

Can you make a tour visiting each bridge exactly once?

"Konigsberg bridges" by Bogdan Giuşcă - License.



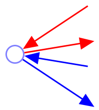
Can you draw a tour in the graph where you visit each edge once?
Yes? No?
We will see!

Eulerian Tour

An Eulerian Tour is a tour that visits each edge exactly once.

Theorem: Any undirected graph has an Eulerian tour if and only if all vertices have even degree and is connected.
Proof of only if: Eulerian $\implies$ connected and all even degree.

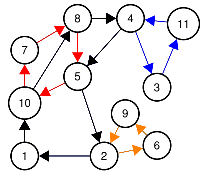
Eulerian Tour is connected so graph is connected.
Tour enters and leaves vertex v on each visit.
Uses two incident edges per visit. Tour uses all incident edges.
Therefore v has even degree. □



When you enter, you can leave.
For starting node, tour leaves first ....then enters at end.
Not The Hotel California.

Finding a tour!

Proof of if: Even + connected $\implies$ Eulerian Tour.
We will give an algorithm. First by picture.



1. Take a walk starting from v (1) on "unused" edges
   ... till you get back to v.
2. Remove tour, C.
3. Let $G_1, \ldots, G_k$ be connected components.
   Each is touched by C.
   Why? G was connected.
   Let v_i be (first) node in G_i touched by C.
   Example: $v_1 = 1$, $v_2 = 10$, $v_3 = 4$, $v_4 = 2$.
4. Recurse on $G_1, \ldots, G_k$ starting from v_i
5. Splice together.
   1,10,7,8,5,10 ,8,4,3,11,4 5,2,6,9,2 and to 1!

Recursive/Inductive Algorithm.

1. Take a walk from arbitrary node v, until you get back to v.

Claim: Do get back to v!
Proof of Claim: Even degree. If enter, can leave except for v. □

2. Remove cycle, C, from G.

Resulting graph may be disconnected. (Removed edges!)
Let components be $G_1, \ldots, G_k$.
Let v_i be first vertex of C that is in G_i.
Why is there a v_i in C?
G was connected $\Longrightarrow$
a vertex in G_i must be incident to a removed edge in C.

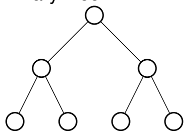
Claim: Each vertex in each G_i has even degree and is connected.
Prf: Tour C has even incidences to any vertex v. □

3. Find tour T_i of G_i starting/ending at v_i. Induction.
4. Splice T_i into C where v_i first appears in C.

Visits every edge once:
Visits edges in C exactly once.
By induction for all edges in each G_i. □

A Tree, a tree.

Graph $G = (V, E)$.
Binary Tree!



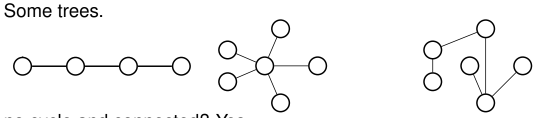
More generally.

Trees.

Definitions:

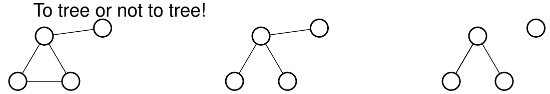
A connected graph without a cycle.
A connected graph with $|V| - 1$ edges.
A connected graph where any edge removal disconnects it.
A connected graph where any edge addition creates a cycle.

Some trees.



no cycle and connected? Yes.
$|V| - 1$ edges and connected? Yes.
removing any edge disconnects it. Harder to check. but yes.
Adding any edge creates cycle. Harder to check. but yes.

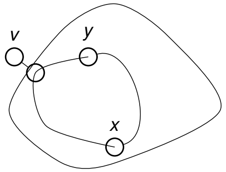
To tree or not to tree!



Equivalence of Definitions.

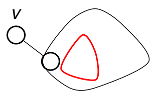
Theorem:
"G connected and has $|V| - 1$ edges" $\equiv$
"G is connected and has no cycles."

Lemma: If v is degree 1 in connected graph G, $G - v$ is connected.
Proof:
For $x \neq v, y \neq v \in V$,
there is path between x and y in G since connected.
and does not use v (degree 1)
$\Longrightarrow$ $G - v$ is connected. □



Proof of only if.

Thm:
"G connected and has $|V| - 1$ edges" $\Longrightarrow$
"G is connected and has no cycles."



Proof of $\Longrightarrow$: By induction on $|V|$.
Base Case: $|V| = 1$. $0 = |V| - 1$ edges and has no cycles.

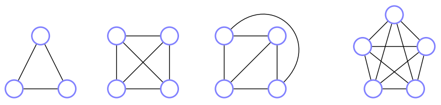
Induction Step:
Claim: There is a degree 1 node.
Proof: First, connected $\Longrightarrow$ every vertex degree ≥ 1.
Sum of degrees is $2|E| = 2(|V| - 1) = 2|V| - 2$
Average degree $2 - 2/|V|$
Not everyone is bigger than average! □
By degree 1 removal lemma, $G - v$ is connected.
$G - v$ has $|V| - 1$ vertices and $|V| - 2$ edges so by induction
$\Longrightarrow$ no cycle in $G - v$.
And no cycle in G since degree 1 cannot participate in cycle. □

Proof of if

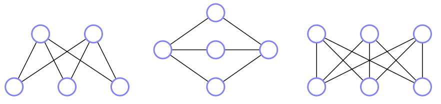
Thm:
"G is connected and has no cycles"
$\Longrightarrow$ "G connected and has $|V| - 1$ edges"
Proof:
Walk from a vertex using untraversed edges.
Until get stuck.
Claim: Degree 1 vertex.
Proof of Claim:
Can't visit more than once since no cycle.
Entered. Didn't leave. Only one incident edge. □
Removing node doesn't create cycle.
New graph is connected.
Removing degree 1 node doesn't disconnect from Degree 1 lemma.
By induction $G - v$ has $|V| - 2$ edges.
G has one more or $|V| - 1$ edges. □

Planar graphs.

A graph that can be drawn in the plane without edge crossings.



Planar? Yes for Triangle.
Four node complete? Yes.
(complete ≡ every edge present. K_n is n-vertex complete graph.)
Five node complete or K_5 ? No! Why? Later.



Two to three nodes, bipartite? Yes.
Three to three nodes, complete/bipartite or $K_{3,3}$. No. Why? Later.