Today

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Finish Euclid.

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Bijection/CRT/Isomorphism.

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Fermat's Little Theorem.

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Excursion: Value and Size.

Before discussing running time of gcd procedure...

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Algorithms at work.

Trying everything

Algorithms at work.

Trying everything

Check 2, check 3, check 4, check 5 . . . , check $y/2$.

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Check 2, check 3, check 4, check 5 . . . , check $y/2$.

“(gcd x y)” at work.

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Check 2, check 3, check 4, check 5 . . . , check $y/2$.

"(gcd x y)" at work.

```
euclid(700,568)
```

Algorithms at work.

Trying everything

Check 2, check 3, check 4, check 5 . . . , check $y/2$.

"(gcd x y)" at work.

```
euclid(700,568)
  euclid(568, 132)
```

Algorithms at work.

Trying everything

Check 2, check 3, check 4, check 5 . . . , check $y/2$.

"(gcd x y)" at work.

```
euclid(700,568)
  euclid(568, 132)
    euclid(132, 40)
```

Algorithms at work.

Trying everything

Check 2, check 3, check 4, check 5 . . . , check $y/2$.

"(gcd x y)" at work.

```
euclid(700,568)
  euclid(568, 132)
    euclid(132, 40)
      euclid(40, 12)
```

Algorithms at work.

Trying everything

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  euclid(568, 132)
    euclid(132, 40)
      euclid(40, 12)
        euclid(12, 4)
```

Algorithms at work.

Trying everything

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"(gcd x y)" at work.

```
euclid(700,568)
  euclid(568, 132)
    euclid(132, 40)
      euclid(40, 12)
        euclid(12, 4)
          euclid(4, 0)
```

Algorithms at work.

Trying everything

Check 2, check 3, check 4, check 5 . . . , check $y/2$.

"(gcd x y)" at work.

```
euclid(700,568)
  euclid(568, 132)
    euclid(132, 40)
      euclid(40, 12)
        euclid(12, 4)
          euclid(4, 0)
            4
```

Algorithms at work.

Trying everything

Check 2, check 3, check 4, check 5 . . . , check $y/2$.

“(gcd x y)” at work.

```
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  euclid(568, 132)
    euclid(132, 40)
      euclid(40, 12)
        euclid(12, 4)
          euclid(4, 0)
            4
```

Notice: The first argument decreases rapidly.

Algorithms at work.

Trying everything

Check 2, check 3, check 4, check 5 . . . , check $y/2$.

“(gcd x y)” at work.

```
euclid(700,568)
  euclid(568, 132)
    euclid(132, 40)
      euclid(40, 12)
        euclid(12, 4)
          euclid(4, 0)
            4
```

Notice: The first argument decreases rapidly.
 At least a factor of 2 in two recursive calls.

Algorithms at work.

Trying everything

Check 2, check 3, check 4, check 5 . . . , check $y/2$.

"(gcd x y)" at work.

```
euclid(700,568)
  euclid(568, 132)
    euclid(132, 40)
      euclid(40, 12)
        euclid(12, 4)
          euclid(4, 0)
            4
```

Notice: The first argument decreases rapidly.
At least a factor of 2 in two recursive calls.

(The second is less than the first.)

Poll.

Runtime Proof.

```
(define (euclid x y)
 (if (= y 0)
     x
    (euclid  y (mod x y))))
```

Theorem: (euclid x y) uses $O(n)$ "divisions" where $n = b(x)$.

Runtime Proof.

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(define (euclid x y)
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Theorem: (euclid x y) uses $O(n)$ ”divisions” where $n = b(x)$.

Proof:

Fact:
First arg decreases by at least factor of two in two recursive calls.

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Proof:

Fact:
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After $2\log_2 x = O(n)$ recursive calls, argument x is 1 bit number.

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One more recursive call to finish.

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After $2\log_2 x = O(n)$ recursive calls, argument x is 1 bit number.
One more recursive call to finish.
1 division per recursive call.

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Proof:

Fact:
First arg decreases by at least factor of two in two recursive calls.

After $2\log_2 x = O(n)$ recursive calls, argument x is 1 bit number.
One more recursive call to finish.
1 division per recursive call.
$O(n)$ divisions.

□

Runtime Proof (continued.)

```
(define (euclid x y)
 (if (= y 0)
     x
    (euclid  y (mod x y))))
```

Fact:
First arg decreases by at least factor of two in two recursive calls.

Runtime Proof (continued.)

```
(define (euclid x y)
 (if (= y 0)
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Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Runtime Proof (continued.)

```
(define (euclid x y)
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Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: $y < x/2$, first argument is y
$\implies$ true in one recursive call;

Runtime Proof (continued.)

```
(define (euclid x y)
 (if (= y 0)
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Fact:
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Proof of Fact: Recall that first argument decreases every call.

Case 1: $y < x/2$, first argument is y
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Case 2: Will show "$y \geq x/2$" $\implies$ "$mod(x, y) \leq x/2$."

Runtime Proof (continued.)

```
(define (euclid x y)
 (if (= y 0)
     x
    (euclid  y (mod x y))))
```

Fact:
First arg decreases by at least factor of two in two recursive calls.

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mod (x, y) is second argument in next recursive call,

Runtime Proof (continued.)

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(define (euclid x y)
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mod (x,y) is second argument in next recursive call,
and becomes the first argument in the next one.

Runtime Proof (continued.)

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mod (x,y) is second argument in next recursive call,
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When $y \geq x/2$, then

Runtime Proof (continued.)

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mod (x, y) is second argument in next recursive call,
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When $y \geq x/2$, then

$\lfloor \frac{x}{y} \rfloor = 1$,

Runtime Proof (continued.)

```
(define (euclid x y)
 (if (= y 0)
     x
    (euclid  y (mod x y))))
```

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: $y < x/2$, first argument is y
$\Longrightarrow$ true in one recursive call;

Case 2: Will show "$y \geq x/2$" $\Longrightarrow$ "$mod(x,y) \leq x/2$."

mod (x, y) is second argument in next recursive call,
and becomes the first argument in the next one.
When $y \geq x/2$, then

$\lfloor \frac{x}{y} \rfloor = 1$,

$\text{mod}\,(x,y) = x - y\lfloor \frac{x}{y} \rfloor =$

Runtime Proof (continued.)

```
(define (euclid x y)
 (if (= y 0)
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Fact:
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When $y \geq x/2$, then

$\lfloor \frac{x}{y} \rfloor = 1,$

$\text{mod}\,(x, y) = x - y\lfloor \frac{x}{y} \rfloor = x - y \leq x - x/2$

Runtime Proof (continued.)

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(define (euclid x y)
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Fact:
First arg decreases by at least factor of two in two recursive calls.

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mod (x,y) is second argument in next recursive call,
and becomes the first argument in the next one.

When $y \geq x/2$, then

$\lfloor \frac{x}{y} \rfloor = 1$,

mod $(x,y) = x - y\lfloor \frac{x}{y} \rfloor = x - y \leq x - x/2 = x/2$

Runtime Proof (continued.)

```
(define (euclid x y)
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     x
    (euclid  y (mod x y))))
```

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: $y < x/2$, first argument is y
$\implies$ true in one recursive call;

Case 2: Will show "$y \geq x/2$" $\implies$ "$mod(x,y) \leq x/2$."

$\bmod(x,y)$ is second argument in next recursive call,
and becomes the first argument in the next one.

When $y \geq x/2$, then

$$\lfloor \frac{x}{y} \rfloor = 1,$$

$$\bmod(x,y) = x - y\lfloor \frac{x}{y} \rfloor = x - y \leq x - x/2 = x/2$$

$\square$

Finding an inverse?

We showed how to efficiently tell if there is an inverse.

Finding an inverse?

We showed how to efficiently tell if there is an inverse.

Extend euclid to find inverse.

Euclid's GCD algorithm.

```
(define (euclid x y)
 (if (= y 0)
     x
    (euclid  y (mod x y))))
```

Euclid's GCD algorithm.

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(define (euclid x y)
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Computes the $\gcd(x, y)$ in $O(n)$ divisions. (Remember $n = \log_2 x$.)

Euclid's GCD algorithm.

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```

Computes the $\gcd(x,y)$ in $O(n)$ divisions. (Remember $n = \log_2 x$.)

For x and m, if $\gcd(x,m) = 1$ then x has an inverse modulo m.

Multiplicative Inverse.

GCD algorithm used to tell **if** there is a multiplicative inverse.

Multiplicative Inverse.

GCD algorithm used to tell **if** there is a multiplicative inverse.

How do we **find** a multiplicative inverse?

Extended GCD

Euclid's Extended GCD Theorem: For any x, y there are integers a, b such that

$$ax + by$$

Extended GCD

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"Make d out of sum of multiples of x and y."

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What is multiplicative inverse of x modulo m?

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What is multiplicative inverse of x modulo m?

By extended GCD theorem, when $\gcd(x, m) = 1$.

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$$ax + bm = 1$$

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By extended GCD theorem, when $\gcd(x, m) = 1$.

$$ax + bm = 1$$
$$ax \equiv 1 - bm \equiv 1 \pmod{m}.$$

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So a multiplicative inverse of $x \pmod{m}$!!

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So a multiplicative inverse of $x \pmod{m}$!!
Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

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Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

$(3)12 + (-1)35 = 1$.

Extended GCD

Euclid's Extended GCD Theorem: For any x, y there are integers a, b such that

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What is multiplicative inverse of x modulo m?

By extended GCD theorem, when $\gcd(x, m) = 1$.

$$ax + bm = 1$$
$$ax \equiv 1 - bm \equiv 1 \pmod{m}.$$

So a multiplicative inverse of $x \pmod{m}$!!
Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

$$(3)12 + (-1)35 = 1.$$

$$a = 3 \text{ and } b = -1.$$

Extended GCD

Euclid's Extended GCD Theorem: For any x, y there are integers a, b such that
$ax + by = d$ where $d = gcd(x, y)$.

"Make d out of sum of multiples of x and y."

What is multiplicative inverse of x modulo m?

By extended GCD theorem, when $\gcd(x, m) = 1$.

$$ax + bm = 1$$
$$ax \equiv 1 - bm \equiv 1 \pmod{m}.$$

So a multiplicative inverse of $x \pmod{m}$!!
Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

$$(3)12 + (-1)35 = 1.$$

$a = 3$ and $b = -1$.
The multiplicative inverse of $12 \pmod{35}$ is 3.

Extended GCD

Euclid's Extended GCD Theorem: For any x, y there are integers a, b such that

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What is multiplicative inverse of x modulo m?

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Check: 3(12)

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$ax + by = d$ where $d = gcd(x, y)$.

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Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

$$(3)12 + (-1)35 = 1.$$

$a = 3$ and $b = -1$.
The multiplicative inverse of $12 \pmod{35}$ is 3.

Check: $3(12) = 36$

Extended GCD

Euclid's Extended GCD Theorem: For any x, y there are integers a, b such that

$$ax + by = d \qquad \text{where } d = gcd(x, y).$$

"Make d out of sum of multiples of x and y."

What is multiplicative inverse of x modulo m?

By extended GCD theorem, when $\gcd(x, m) = 1$.

$$ax + bm = 1$$
$$ax \equiv 1 - bm \equiv 1 \pmod{m}.$$

So a multiplicative inverse of $x \pmod{m}$!!
Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

$$(3)12 + (-1)35 = 1.$$

$a = 3$ and $b = -1$.
The multiplicative inverse of $12 \pmod{35}$ is 3.

Check: $3(12) = 36 = 1 \pmod{35}$.

Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
```

Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
```

Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
```

Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?

Make d out of multiples of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?

Make d out of multiples of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Make d out of multiples of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.

Make d out of multiples of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11$

Make d out of multiples of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
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Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11 = 12 - (1)(35 - (2)12)$
Get 11 from 35 and 12 and plugin....

Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

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But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35$
Get 11 from 35 and 12 and plugin.... Simplify.

Make d out of multiples of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
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But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35$
Get 11 from 35 and 12 and plugin.... Simplify. $a = 3$ and $b = -1$.

Extended GCD Algorithm.

```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
     else
         (d, a, b) := ext-gcd(y, mod(x,y))
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Claim: Returns (d,a,b): $d = gcd(a,b)$ and $d = ax + by$.

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```
ext-gcd(35,12)
  ext-gcd(12, 11)
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Claim: Returns (d,a,b): $d = gcd(a,b)$ and $d = ax + by$.
Example: $a - \lfloor x/y \rfloor \cdot b =$

```
ext-gcd(35,12)
  ext-gcd(12, 11)
    ext-gcd(11, 1)
      ext-gcd(1,0)
      return (1,1,0) ;; 1 = (1)1 + (0) 0
```

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Example: $a - \lfloor x/y \rfloor \cdot b = 1 - \lfloor 11/1 \rfloor \cdot 0 = 1$

```
ext-gcd(35,12)
  ext-gcd(12, 11)
    ext-gcd(11, 1)
      ext-gcd(1,0)
      return (1,1,0) ;; 1 = (1)1 + (0) 0
    return (1,0,1)   ;; 1 = (0)11 + (1)1
```

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Claim: Returns (d, a, b): $d = gcd(a, b)$ and $d = ax + by$.
Example: $a - \lfloor x/y \rfloor \cdot b = \quad 0 - \lfloor 12/11 \rfloor \cdot 1 = -1$

```
ext-gcd(35,12)
  ext-gcd(12, 11)
    ext-gcd(11, 1)
      ext-gcd(1,0)
      return (1,1,0) ;; 1 = (1)1 + (0) 0
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  return (1,1,-1)    ;; 1 = (1)12 + (-1)11
```

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```

Claim: Returns (d,a,b): $d = gcd(a,b)$ and $d = ax + by$.
Example: $a - \lfloor x/y \rfloor \cdot b =$ $1 - \lfloor 35/12 \rfloor \cdot (-1) = 3$

```
    ext-gcd(35,12)
      ext-gcd(12, 11)
        ext-gcd(11, 1)
          ext-gcd(1,0)
          return (1,1,0) ;; 1 = (1)1 + (0) 0
        return (1,0,1)   ;; 1 = (0)11 + (1)1
      return (1,1,-1)    ;; 1 = (1)12 + (-1)11
   return (1,-1, 3)      ;; 1 = (-1)35 +(3)12
```

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      return (1,1,-1)    ;; 1 = (1)12 + (-1)11
   return (1,-1, 3)      ;; 1 = (-1)35 +(3)12
```

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```
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         return (d, b, a - floor(x/y)  * b)
```

Theorem: Returns (d, a, b), where $d = gcd(a, b)$ and

$$d = ax + by.$$

Correctness.

Proof: Strong Induction.[1]

[1] Assume d is $gcd(x, y)$ by previous proof.

Correctness.

Proof: Strong Induction.[1]
Base: ext-gcd($x,0$) returns ($d = x, 1, 0$) with $x = (1)x + (0)y$.

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Ind hyp: **ext-gcd**$(y, \mod(x,y))$ returns (d,a,b) with
$d=ay+b(\mod(x,y))$

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$$d = ay + b \cdot (\mod(x, y))$$

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$$\begin{aligned} d &= ay + b \cdot (\mod(x, y)) \\ &= ay + b \cdot (x - \lfloor \frac{x}{y} \rfloor y) \end{aligned}$$

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$$\begin{aligned} d &= ay + b \cdot (\mod(x,y)) \\ &= ay + b \cdot (x - \lfloor \frac{x}{y} \rfloor y) \\ &= bx + (a - \lfloor \frac{x}{y} \rfloor \cdot b)y \end{aligned}$$

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And ext-gcd returns $(d, b, (a - \lfloor\frac{x}{y}\rfloor\cdot b))$ so theorem holds!

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Review Proof: step.

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```

Recursively: $d = ay + b(x - \lfloor \frac{x}{y} \rfloor \cdot y)$

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ext-gcd(x,y)
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```

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Recursively: $d = ay + b(x - \lfloor \frac{x}{y} \rfloor \cdot y) \implies d = bx - (a - \lfloor \frac{x}{y} \rfloor b)y$

Returns $(d, b, (a - \lfloor \frac{x}{y} \rfloor \cdot b))$.

Hand Calculation Method for Inverses.

Example: $\gcd(7,60) = 1$.

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$$\begin{aligned} 7(0) + 60(1) &= 60 \\ 7(1) + 60(0) &= 7 \end{aligned}$$

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$$\begin{aligned} 7(0) + 60(1) &= 60 \\ 7(1) + 60(0) &= 7 \\ 7(-8) + 60(1) &= 4 \end{aligned}$$

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Confirm:

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Confirm: $-119+120=1$

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Confirm: $-119+120=1$

Note: an “iterative” version of the e-gcd algorithm.

Wrap-up

Conclusion: Can find multiplicative inverses in $O(n)$ time!

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Inverse of $500,000,357$ modulo $1,000,000,000,000$?

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Inverse of $500,000,357$ modulo $1,000,000,000,000$?
≤ 80 divisions.

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Internet Security.

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Public Key Cryptography: 512 digits.
512 divisions vs.

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512 divisions vs.
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Internet Security: Soon.

Bijections

Bijection is **one to one** and **onto.**
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$f : A \to B$.

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Domain: A, Co-Domain: B.

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E.g. $\mathbf{sin}\,(x)$.

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  $A = B =$ reals.

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 Range is $[-1, 1]$.

Bijections

Bijection is **one to one** and **onto.**
Bijection:
  $f : A \to B$.
Domain: A, Co-Domain: B.
 Versus Range.
E.g. **sin** (x).
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Lots of Mods

$x = 5 \pmod 7$ and $x = 3 \pmod 5$.

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Let's try 3.

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Let's try 5. Not $3 \pmod 5$!
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If $x = 5 \pmod 7$
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A bit slow for large values.

Simple Chinese Remainder Theorem.

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Find $x = a \pmod{m}$ and $x = b \pmod{n}$

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Find $x = a \pmod{m}$ and $x = b \pmod{n}$ where gcd(m, n)=1.

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$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$
$x = b \pmod{n}$ since $au = 0 \pmod{n}$ and $bv = b \pmod{n}$

Only solution?

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$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$
$x = b \pmod{n}$ since $au = 0 \pmod{n}$ and $bv = b \pmod{n}$

Only solution? If not, two solutions, x and y.

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$v = 1 \pmod{n}$ $\quad$ $v = 0 \pmod{m}$

Let $x = au + bv$.
$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$
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Only solution? If not, two solutions, x and y.
$(x - y) \equiv 0 \pmod{m}$ and $(x - y) \equiv 0 \pmod{n}$.

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Find $x = a \pmod{m}$ and $x = b \pmod{n}$ where gcd(m, n)=1.

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Consider $v = m(m^{-1} \pmod{n})$.
$v = 1 \pmod{n}$ $\qquad v = 0 \pmod{m}$

Let $x = au + bv$.
$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$
$x = b \pmod{n}$ since $au = 0 \pmod{n}$ and $bv = b \pmod{n}$

Only solution? If not, two solutions, x and y.
$(x - y) \equiv 0 \pmod{m}$ and $(x - y) \equiv 0 \pmod{n}$.
$\implies (x - y)$ is multiple of m and n since gcd(m, n)=1.

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Only solution? If not, two solutions, x and y.
$(x-y) \equiv 0 \pmod{m}$ and $(x-y) \equiv 0 \pmod{n}$.
$\implies (x-y)$ is multiple of m and n since gcd(m,n)=1.
$\implies x-y \geq mn$

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$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$
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Only solution? If not, two solutions, x and y.
$(x - y) \equiv 0 \pmod{m}$ and $(x - y) \equiv 0 \pmod{n}$.
$\implies (x - y)$ is multiple of m and n since gcd(m, n)=1.
$\implies x - y \geq mn \implies x, y \notin \{0, \ldots, mn - 1\}$.

Simple Chinese Remainder Theorem.

My love is won. Zero and One. Nothing and nothing done.

Find $x = a \pmod{m}$ and $x = b \pmod{n}$ where gcd(m,n)=1.

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Proof:
Consider $u = n(n^{-1} \pmod{m})$.
$u = 0 \pmod{n}$ $\quad$ $u = 1 \pmod{m}$

Consider $v = m(m^{-1} \pmod{n})$.
$v = 1 \pmod{n}$ $\quad$ $v = 0 \pmod{m}$

Let $x = au + bv$.
$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$
$x = b \pmod{n}$ since $au = 0 \pmod{n}$ and $bv = b \pmod{n}$

Only solution? If not, two solutions, x and y.
$(x-y) \equiv 0 \pmod{m}$ and $(x-y) \equiv 0 \pmod{n}$.
$\implies (x-y)$ is multiple of m and n since gcd(m,n)=1.
$\implies x - y \geq mn \implies x,y \notin \{0,\ldots,mn-1\}$.
Thus, only one solution modulo mn.

Simple Chinese Remainder Theorem.

My love is won. Zero and One. Nothing and nothing done.

Find $x = a \pmod{m}$ and $x = b \pmod{n}$ where gcd(m,n)=1.

CRT Thm: There is a unique solution $x \pmod{mn}$.
Proof:
Consider $u = n(n^{-1} \pmod{m})$.
$u = 0 \pmod{n}$ $\qquad u = 1 \pmod{m}$

Consider $v = m(m^{-1} \pmod{n})$.
$v = 1 \pmod{n}$ $\qquad v = 0 \pmod{m}$

Let $x = au + bv$.
$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$
$x = b \pmod{n}$ since $au = 0 \pmod{n}$ and $bv = b \pmod{n}$

Only solution? If not, two solutions, x and y.
$(x - y) \equiv 0 \pmod{m}$ and $(x - y) \equiv 0 \pmod{n}$.
$\implies (x - y)$ is multiple of m and n since gcd(m,n)=1.
$\implies x - y \geq mn \implies x, y \notin \{0, \ldots, mn - 1\}$.
Thus, only one solution modulo mn. $\square$

CRT:isomorphism.

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Mapping is "isomorphic":
corresponding addition (and multiplication) operations consistent with mapping.

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For a prime modulus, we can reduce exponents modulo $p-1$!

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Product of elts == for range/domain: a^{p-1} factor in range.