CS 70 Discrete Mathematics and Probability Theory
Fall 2020

HW 11

Due: Friday 11/14 at 10:00 PM
Grace period until Friday 11/14 at 11:59PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Random Variables Warm-Up

Let X and Y be random variables, each taking values in the set $\{0,1,2\}$, with joint distribution

$$\begin{array}{lll}
\mathbb{P}[X=0,Y=0]=1/3 & \mathbb{P}[X=0,Y=1]=0 & \mathbb{P}[X=0,Y=2]=1/3 \\
\mathbb{P}[X=1,Y=0]=0 & \mathbb{P}[X=1,Y=1]=1/9 & \mathbb{P}[X=1,Y=2]=0 \\
\mathbb{P}[X=2,Y=0]=1/9 & \mathbb{P}[X=2,Y=1]=1/9 & \mathbb{P}[X=2,Y=2]=0.
\end{array}$$

(a) What are the marginal distributions of X and Y?

(b) What are $\mathbb{E}[X]$ and $\mathbb{E}[Y]$?

(c) (optional) What are $\mathrm{Var}(X)$ and $\mathrm{Var}(Y)$?

(d) Let I be the indicator that $X=1$, and J be the indicator that $Y=1$. What are $\mathbb{E}[I]$, $\mathbb{E}[J]$ and $\mathbb{E}[IJ]$?

(e) In general, let I_A and I_B be the indicators for events A and B in a probability space $(\Omega,\mathbb{P})$. What is $\mathbb{E}[I_A I_B]$, in terms of the probability of some event?

2 Marginals

(a) Can there exist three random variables X_1,X_2,X_3, each taking values in the set $\{+1,-1\}$, with the property that for every $i \neq j$, the joint distribution of X_i and X_j is given by

$$\mathbb{P}[X_i=1,X_j=-1]=\frac{1}{2} \qquad \mathbb{P}[X_i=-1,X_j=1]=\frac{1}{2} \qquad \mathbb{P}[X_i=X_j]=0? \tag{1}$$

If so, specify the joint distribution of X_1,X_2,X_3; if not, prove it.

(b) For which natural numbers $n \geq 3$ can there exist random variables $X_1, X_2, ..., X_n$, each taking values in the set $\{+1, -1\}$, with the property that for every i and j satisfying $i - j = 1 \pmod{n}$, the joint distribution of X_i and X_j is given by (1)? For any n that work, specify the joint distribution; for those that do not, prove it.

3 Testing Model Planes

Amin is testing model airplanes. He starts with n model planes which each independently have probability p of flying successfully each time they are flown, where $0 < p < 1$. Each day, he flies every single plane and keeps the ones that fly successfully (i.e. don't crash), throwing away all other models. He repeats this process for many days, where each "day" consists of Amin flying any remaining model planes and throwing away any that crash. Let X_i be the random variable representing how many model planes remain after i days. Note that $X_0 = n$. Justify your answers for each part.

(a) What is the distribution of X_1? That is, what is $\mathbb{P}[X_1 = k]$?

(b) What is the distribution of X_2? That is, what is $\mathbb{P}[X_2 = k]$? Name the distribution of X_2 and what its parameters are.

(c) Repeat the previous part for X_t for arbitrary $t \geq 1$.

(d) What is the probability that at least one model plane still remains (has not crashed yet) after t days? Do not have any summations in your answer.

(e) Considering only the first day of flights, is the event A_1 that the first and second model planes crash independent from the event B_1 that the second and third model planes crash? Recall that two events A and B are independent if $\mathbb{P}[A \cap B] = \mathbb{P}[A]\mathbb{P}[B]$. Prove your answer using this definition.

(f) Considering only the first day of flights, let A_2 be the event that the first model plane crashes *and* exactly two model planes crash in total. Let B_2 be the event that the second plane crashes on the first day. What must n be equal to in terms of p such that A_2 is independent from B_2? Prove your answer using the definition of independence stated in the previous part.

(g) Are the random variables X_i and X_j, where $i < j$, independent? Recall that two random variables X and Y are independent if $\mathbb{P}[X = k_1 \cap Y = k_2] = \mathbb{P}[X = k_1]\mathbb{P}[Y = k_2]$ for all k_1 and k_2. Prove your answer using this definition.

4 Graph

Consider a random graph (undirected, no multi-edges, no self-loops) on n nodes, where each possible edge exists independently with probability p. Let X be the number of isolated nodes (nodes with degree 0).

(a) What is $E(X)$? *Consider X to be the sum of the indicators X_i that vertex i is isolated. Why isn't X a binomial random variable?*

(b) (optional) What is $\text{Var}(X)$?

5 Triangles in Random Graphs

Let's say we make a simple and undirected graph G on n vertices by randomly adding m edges, without replacement. In other words, we choose the first edge uniformly from all $\binom{n}{2}$ possible edges, then the second one uniformly from among the remaining $\binom{n}{2} - 1$ edges, etc. What is the expected number of triangles in G? (A triangle is a triplet of distinct vertices with all three edges present between them.)