CS 70 Discrete Mathematics and Probability Theory
Fall 2020

HW 13

Due: Monday 11/30 2020 at 10:00PM
Grace period until Monday 11/30 2020 at 11:59PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Random Cuckoo Hashing

Cuckoo birds are parasitic beasts. They are known for hijacking the nests of other bird species and evicting the eggs already inside. Cuckoo hashing is inspired by this behavior. In cuckoo hashing, when we get a collision, the element that was already there gets evicted and rehashed.

We study a simple (but ineffective, as we'll see) version of cuckoo hashing, where all hashes are random. Let's say we want to hash n pieces of data $d_1, d_2, \ldots, d_n$ into n possible hash buckets labeled $1, \ldots, n$. We hash the $d_1, \ldots, d_n$ in that order. When hashing d_i, we assign it a random bucket chosen uniformly from $1, \ldots, n$. If there is no collision, then we place d_i into that bucket. If there is a collision with some other d_j, we evict d_j and assign it another random bucket uniformly from $1, \ldots, n$. (It is possible that d_j gets assigned back to the bucket it was just evicted from!) We again perform the eviction step if we get another collision. We keep doing this until there is no more collision, and we then introduce the next piece of data, d_{i+1} to the hash table.

(a) What is the probability that there are no collisions over the entire process of hashing $d_1, \ldots, d_n$ to buckets $1, \ldots, n$? What value does the probability tend towards as n grows very large?

(b) Assume we have already hashed $d_1, \ldots, d_{n-1}$, and they each occupy their own bucket. We now introduce d_n into our hash table. What is the expected number of collisions that we'll see while hashing d_n? (*Hint*: What happens when we hash d_n and get a collision, so we evict some other d_i and have to hash d_i? Are we at a situation that we've seen before?)

(c) Generalize the previous part: Assume we have already hashed $d_1, \ldots, d_{k-1}$ successully, where $1 \leq k \leq n$. Let C_k be the number of collisions that we'll see while hashing d_k. What is $\mathbb{E}[C_k]$?

(d) Let C be the total number of collisions over the entire process of hashing $d_1, \ldots, d_n$. What is $\mathbb{E}[C]$? You may leave your answer as a summation.

2 Geometric and Poisson

Let $X \sim \text{Geo}(p)$ and $Y \sim \text{Poisson}(\lambda)$ be independent. random variables. Compute $\mathbb{P}(X > Y)$. Your final answer should not have summations.

3 Exploring the Geometric Distribution

Suppose $X \sim \text{Geometric}(p)$ and $Y \sim \text{Geometric}(q)$ are independent. Find the distribution of $\min\{X, Y\}$ and justify your answer.

4 Lunch Meeting

Alice and Bob agree to try to meet for lunch between 12 PM and 1 PM at their favorite sushi restaurant. Being extremely busy, they are unable to specify their arrival times exactly, and can say only that each of them will arrive (independently) at a time that is uniformly distributed within the hour. In order to avoid wasting precious time, if the other person is not there when they arrive they agree to wait exactly fifteen minutes before leaving. What is the probability that they will actually meet for lunch? (hint: Sketch the joint distribution of the arrival times of Alice and Bob. What parts of the distribution corresponds to them meeting for lunch?)