Due: Sunday 12/06 2020 at 10:00 PM
Grace period until Sunday 12/06 2020 at 11:59PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Short Answer

(a) Let X be uniform on the interval $[0,2]$, and define $Y = 2X+1$. Find the PDF, CDF, expectation, and variance of Y.

(b) Let X and Y have joint distribution

$$f(x,y) = \begin{cases} cxy + 1/4 & x \in [1,2] \text{ and } y \in [0,2] \\ 0 & \text{else} \end{cases}$$

Find the constant c. Are X and Y independent?

2 Continuous Probability Continued

For the following questions, please briefly justify your answers or show your work.

(a) Assume $\mathrm{Bob}_1, \mathrm{Bob}_2, \ldots, \mathrm{Bob}_k$ each hold a fair coin whose two sides show numbers instead of heads and tails, with the numbers on Bob_i's coin being i and $-i$. Each Bob tosses their coin n times and sums up the numbers he sees; let's call this number X_i. For large n, what is the distribution of $(X_1 + \cdots + X_k)/\sqrt{n}$ approximately equal to?

(b) If $X_1, X_2, \ldots$ is a sequence of i.i.d. random variables of mean μ and variance σ^2, what is $\lim_{n\to\infty} \mathbb{P}\left[\sum_{k=1}^{n} \frac{X_k - \mu}{\sigma n^{\alpha}} \in [-1,1]\right]$ for $\alpha \in [0,1]$ (your answer may depend on α and Φ, the CDF of a $N(0,1)$ variable)?

3 Exponential Distributions: Lightbulbs

A brand new lightbulb has just been installed in our classroom, and you know the life span of a lightbulb is exponentially distributed with a mean of 50 days.

(a) Suppose an electrician is scheduled to check on the lightbulb in 30 days and replace it if it is broken. What is the probability that the electrician will find the bulb broken?

(b) Suppose the electrician finds the bulb broken and replaces it with a new one. What is the probability that the new bulb will last at least 30 days?

(c) Suppose the electrician finds the bulb in working condition and leaves. What is the probability that the bulb will last at least another 30 days?

4 Useful Uniforms

Let X be a continuous random variable whose image is all of $\mathbb{R}$; that is, $\mathbb{P}[X \in (a,b)] > 0$ for all $a,b \in \mathbb{R}$ and $a \neq b$.

(a) Give an example of a distribution that X could have, and one that it could not.

(b) Show that the CDF F of X is strictly increasing. That is, $F(x+\varepsilon) > F(x)$ for any $\varepsilon > 0$. Argue why this implies that $F : \mathbb{R} \to (0,1)$ must be invertible.

(c) Let U be a uniform random variable on $(0,1)$. What is the distribution of $F^{-1}(U)$?

(d) Your work in part (c) shows that in order to sample X, it is enough to be able to sample U. If X was a discrete random variable instead, taking finitely many values, can we still use U to sample X?

5 Uniform Means

To keep the doctor away, Bob goes to the supermarket to buy an apple. Let $X_1, X_2, \ldots, X_n$ be n independent and identically distributed uniform random variables on the interval $[0,1]$ (where n is a positive integer), where X_i is the quality of the ith apple Bob sees.

(a) Let $Y = \min\{X_1, X_2, \ldots, X_n\}$ be the quality of the worst apple Bob will see. Find $\mathbb{E}(Y)$. [*Hint*: Use the tail sum formula, which says the expected value of a nonnegative random variable is $\mathbb{E}(X) = \int_0^\infty \mathbb{P}(X > x)\,\mathrm{d}x$. Note that we can use the tail sum formula since $Y \geq 0$.]

(b) Let $Z = \max\{X_1, X_2, \ldots, X_n\}$ be the quality of the best apple Bob will see. Find $\mathbb{E}(Z)$. [*Hint*: Find the CDF.]

6 Darts but with ML

Suppose Alice and Bob are playing darts on a circular board with radius 1. When Alice throws a dart, the distance of the dart from the center is uniform $[0,1]$. When Bob throws the dart, the location of the dart is uniform over the whole board. Let X be the random variable corresponding to the distance of the player's dart from the center of the board.

(a) What is the pdf of X if Alice throws

(b) What is the pdf of X if Bob throws

(c) Suppose we let Alice throw the dart with probability p, and let Bob throw otherwise. What is the pdf of X (your answer should be in terms of p)?

(d) Using the same premise as in part c, suppose you observe a dart on the board but don't know who threw it. Let x be the dart's distance from the center. We would like to come up with a decision rule to determine whether Alice or Bob is more likely to have thrown the dart given your observation, x. Specifically, if we let A be the event that Alice threw the dart and B be the event that Bob threw, we want to guess A if $\mathbb{P}[A|X \in [x,x+dx]] > \mathbb{P}[B|X \in [x,x+dx]]$ (what do these two probabilities have to sum up to?). For what values of x would we guess A? (your answer should be in terms of p)

7 Sampling a Gaussian With Uniform

In this question, we will see one way to generate a normal random variable if we have access to a random number generator that outputs numbers between 0 and 1 uniformly at random.

As a general comment, remember that showing two random variables have the same CDF or PDF is sufficient for showing that they have the same distribution.

(a) First, let us see how to generate an exponential random variable with a uniform random variable. Let $U_1 \sim Uniform(0,1)$. Prove that $-\ln U_1 \sim Expo(1)$.

(b) Let $N_1, N_2 \sim \mathcal{N}(0,1)$, where N_1 and N_2 are independent. Prove that $N_1^2 + N_2^2 \sim Expo(1/2)$.

Hint: You may use the fact that over a region R, if we convert to polar coordinates $(x,y) \to (r,\theta)$, then the double integral over the region R will be

$$\iint_R f(x,y)\,dx\,dy = \iint_R f(r\cos\theta, r\sin\theta)\cdot r\,dr\,d\theta.$$

(c) Suppose we have two uniform random variables, U_1 and U_2. How would you transform these two random variables into a normal random variable with mean 0 and variance 1?

Hint: What part (b) tells us is that the point (N_1, N_2) will have a distance from the origin that is distributed as the square root of an exponential distribution. Try to use U_1 to sample the radius, and then use U_2 to sample the angle.