Today.

Polynomials.

Secret Sharing.

Correcting for loss or even corruption.

Secret Sharing.

Share secret among n people.

Secrecy: Any $k-1$ knows nothing.
Roubustness: Any k knows secret.
Efficient: minimize storage.

The idea of the day.

Two points make a line.
Lots of lines go through one point.

Polynomials

A **polynomial**

$$P(x) = a_d x^d + a_{d-1} x^{d-1} \cdots + a_0.$$

is specified by **coefficients** $a_d, \ldots a_0$.

$P(x)$ **contains** point (a,b) if $b = P(a)$.

Polynomials over reals: $a_1, \ldots, a_d \in \Re$, use $x \in \Re$.

Polynomials $P(x)$ **with arithmetic modulo** p: [1] $a_i \in \{0, \ldots, p-1\}$ and

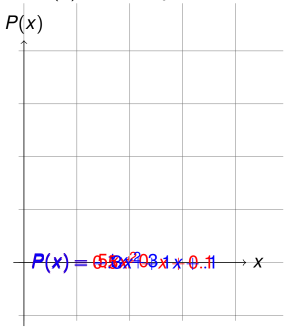
$$P(x) = a_d x^d + a_{d-1} x^{d-1} \cdots + a_0 \pmod p,$$

for $x \in \{0, \ldots, p-1\}$.

[1] A field is a set of elements with addition and multiplication operations, with inverses. $GF(p) = (\{0, \ldots, p-1\}, + \pmod p, * \pmod p)$.

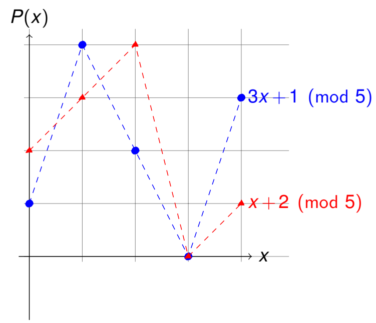
Polynomial: $P(x) = a_d x^4 + \cdots + a_0$

Line: $P(x) = a_1 x + a_0 = mx + b$



Parabola: $P(x) = a_2 x^2 + a_1 x + a_0 = ax^2 + bx + c$

Polynomial: $P(x) = a_d x^4 + \cdots + a_0 \pmod p$



Finding an intersection.
$x + 2 \equiv 3x + 1 \pmod 5$
$\implies 2x \equiv 1 \pmod 5 \implies x \equiv 3 \pmod 5$
3 is multiplicative inverse of 2 modulo 5.
Good when modulus is prime!!

Two points make a line.

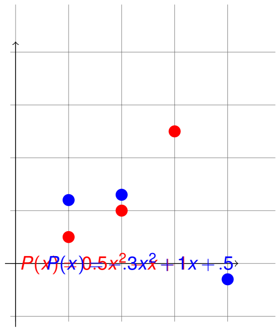
Fact: Exactly 1 degree $\leq d$ polynomial contains $d+1$ points. [2]

Two points specify a line. Three points specify a parabola.

Modular Arithmetic Fact: Exactly 1 degree $\leq d$ polynomial with arithmetic modulo prime p contains $d+1$ pts.

[2] Points with different x values.

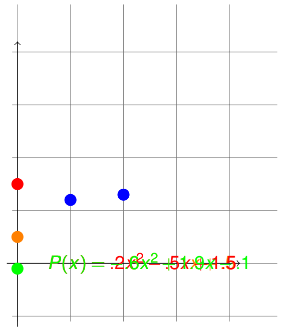
3 points determine a parabola.



Fact: Exactly 1 degree $\leq d$ polynomial contains $d+1$ points. [3]

[3] Points with different x values.

2 points not enough.



There is $P(x)$ contains blue points and *any* $(0,y)$!

Modular Arithmetic Fact and Secrets

Modular Arithmetic Fact: Exactly 1 degree $\leq d$ polynomial with arithmetic modulo prime p contains $d+1$ pts.

Shamir's k out of n Scheme:
Secret $s \in \{0,\ldots,p-1\}$

1. Choose $a_0 = s$, and random $a_1,\ldots,a_{k-1}$.
2. Let $P(x) = a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots a_0$ with $a_0 = s$.
3. Share i is point $(i, P(i) \mod p)$.

Roubustness: Any k shares gives secret.
Knowing k pts $\implies$ only one $P(x)$ $\implies$ evaluate $P(0)$.
Secrecy: Any $k-1$ shares give nothing.
Knowing $\leq k-1$ pts $\implies$ any $P(0)$ is possible.

From $d+1$ points to degree d polynomial?

For a line, $a_1x + a_0 = mx + b$ contains points $(1,3)$ and $(2,4)$.

$$\begin{aligned} P(1) = m(1)+b &\equiv m+b \equiv 3 \pmod 5 \\ P(2) = m(2)+b &\equiv 2m+b \equiv 4 \pmod 5 \end{aligned}$$

Subtract first from second..

$$\begin{aligned} m+b &\equiv 3 \pmod 5 \\ m &\equiv 1 \pmod 5 \end{aligned}$$

Backsolve: $b \equiv 2 \pmod 5$. Secret is 2.

And the line is...

$$x+2 \mod 5.$$

Quadratic

For a quadratic polynomial, $a_2x^2 + a_1x + a_0$ hits $(1,2);(2,4);(3,0)$.
Plug in points to find equations.

$$\begin{aligned} P(1) = a_2 + a_1 + a_0 &\equiv 2 \pmod 5 \\ P(2) = 4a_2 + 2a_1 + a_0 &\equiv 4 \pmod 5 \\ P(3) = 4a_2 + 3a_1 + a_0 &\equiv 0 \pmod 5 \end{aligned}$$

$$\begin{aligned} a_2 + a_1 + a_0 &\equiv 2 \pmod 5 \\ 3a_1 + 2a_0 &\equiv 1 \pmod 5 \\ 4a_1 + 2a_0 &\equiv 2 \pmod 5 \end{aligned}$$

Subtracting 2nd from 3rd yields: $a_1 = 1$.
$a_0 = (2 - 4(a_1))2^{-1} = (-2)(2^{-1}) = (3)(3) = 9 \equiv 4 \pmod 5$
$a_2 = 2 - 1 - 4 \equiv 2 \pmod 5$.

So polynomial is $2x^2 + 1x + 4 \pmod 5$

In general..

Given points: $(x_1,y_1);(x_2,y_2)\cdots(x_k,y_k)$.

Solve...

$$\begin{aligned} a_{k-1}x_1^{k-1} + \cdots + a_0 &\equiv y_1 \pmod p \\ a_{k-1}x_2^{k-1} + \cdots + a_0 &\equiv y_2 \pmod p \\ &\cdot \\ &\cdot \\ a_{k-1}x_k^{k-1} + \cdots + a_0 &\equiv y_k \pmod p \end{aligned}$$

Will this always work?

As long as solution **exists** and it is **unique!** And...

Modular Arithmetic Fact: Exactly 1 degree $\leq d$ polynomial with arithmetic modulo prime p contains $d+1$ pts.

Another Construction: Interpolation!

For a quadratic, $a_2x^2 + a_1x + a_0$ hits $(1,3);(2,4);(3,0)$.

Find $\Delta_1(x)$ polynomial contains $(1,1);(2,0);(3,0)$.

Try $(x-2)(x-3) \pmod 5$.

Value is 0 at 2 and 3. Value is 2 at 1. Not 1! Doh!!
So "Divide by 2" or multiply by 3.
$\Delta_1(x) = (x-2)(x-3)(3) \pmod 5$ contains $(1,1);(2,0);(3,0)$.

$\Delta_2(x) = (x-1)(x-3)(4) \pmod 5$ contains (1,0);(2,1);(3,0).

$\Delta_3(x) = (x-1)(x-2)(3) \pmod 5$ contains (1,0);(2,0);(3,1).

But wanted to hit $(1,3);(2,4);(3,0)$!

$P(x) = 3\Delta_1(x) + 4\Delta_2(x) + 0\Delta_3(x)$ works.

Same as before?

...after a lot of calculations... $P(x) = 2x^2 + 1x + 4 \mod 5$.

The same as before!

Fields.. .

Flowers, and grass, oh so nice.

Set and two commutative operations: addition and multiplication with additive/multiplicative identities and inverses expect for additive identity has no mulitplicative inverse.

E.g., Reals, rationals, complex numbers.
Not E.g., the integers, matrices.

We will work with polynomials with arithmetic modulo p.

Addition is cool. Inherited from integers and integer division (remainders).
Multiplicative inverses due to $gcd(x,p) = 1$, forall $x \in \{1,\ldots,p-1\}$

Delta Polynomials: Concept.

For set of x-values, $x_1,\ldots,x_{d+1}$.

$$\Delta_i(x) = \begin{cases} 1, & \text{if } x = x_i. \\ 0, & \text{if } x = x_j \text{ for } j \neq i. \\ ?, & \text{otherwise.} \end{cases} \tag{1}$$

Given $d+1$ points, use Δ_i functions to go through points?
$(x_1,y_1),\ldots,(x_{d+1},y_{d+1})$.

Will $y_1\Delta_1(x)$ contain (x_1,y_1)?

Will $y_2\Delta_2(x)$ contain (x_2,y_2)?

Does $y_1\Delta_1(x) + y_2\Delta_2(x)$ contain
(x_1,y_1)? and (x_2,y_2)?

See the idea? Function that contains all points?

$P(x) = y_1\Delta_1(x) + y_2\Delta_2(x) \ldots + y_{d+1}\Delta_{d+1}(x)$.

There exists a polynomial...

Modular Arithmetic Fact: Exactly 1 degree $\leq d$ polynomial with arithmetic modulo prime p contains $d+1$ pts.

Proof of at least one polynomial:
Given points: $(x_1,y_1);(x_2,y_2)\cdots(x_{d+1},y_{d+1})$.

$$\Delta_i(x) = \frac{\prod_{j\neq i}(x-x_j)}{\prod_{j\neq i}(x_i-x_j)} = \prod_{j\neq i}(x-x_j)\prod_{j\neq i}(x_i-x_j)^{-1}$$

Numerator is 0 at $x_j \neq x_i$.

"Denominator" makes it 1 at x_i.

And..

$$P(x) = y_1\Delta_1(x) + y_2\Delta_2(x) + \cdots + y_{d+1}\Delta_{d+1}(x).$$

hits points $(x_1,y_1);(x_2,y_2)\cdots(x_{d+1},y_{d+1})$. Degree d polynomial!

Construction proves the existence of a polynomial!

Example.

$\Delta_i(x) = \frac{\prod_{j\neq i}(x-x_j)}{\prod_{j\neq i}(x_i-x_j)}$.

Degree 1 polynomial, $P(x)$, that contains $(1,3)$ and $(3,4)$?

Work modulo 5.

$\Delta_1(x)$ contains $(1,1)$ and $(3,0)$.

$\Delta_1(x) = \frac{(x-3)}{1-3} = \frac{x-3}{-2}$
$= 2(x-3) = 2x-6 = 2x+4 \pmod 5$.

For a quadratic, $a_2x^2 + a_1x + a_0$ hits $(1,3);(2,4);(3,0)$.

Work modulo 5.

Find $\Delta_1(x)$ polynomial contains $(1,1);(2,0);(3,0)$.

$\Delta_1(x) = \frac{(x-2)(x-3)}{(1-2)(1-3)} = \frac{(x-2)(x-3)}{2} = (2)^{-1}(x-2)(x-3) = 3(x-2)(x-3)$
$= 3x^2 + 3 \pmod 5$

Put the delta functions together.

In general.

Given points: $(x_1,y_1);(x_2,y_2)\cdots(x_k,y_k)$.

$$\Delta_i(x) = \frac{\prod_{j\neq i}(x-x_j)}{\prod_{j\neq i}(x_i-x_j)} = \prod_{j\neq i}(x-x_j)\prod_{j\neq i}(x_i-x_j)^{-1}$$

Numerator is 0 at $x_j \neq x_i$.

Denominator makes it 1 at x_i.

And..

$$P(x) = y_1\Delta_1(x) + y_2\Delta_2(x) + \cdots + y_k\Delta_k(x).$$

hits points $(x_1,y_1);(x_2,y_2)\cdots(x_k,y_k)$.

Construction proves the existence of the polynomial!

Uniqueness.

Uniqueness Fact. At most one degree d polynomial hits $d+1$ points.

Roots fact: Any nontrivial degree d polynomial has at most d roots.

Non-zero line (degree 1 polynomial) can intersect $y=0$ at only one x.

A parabola (degree 2), can intersect $y=0$ at only two x's.

Proof:
Assume two different polynomials $Q(x)$ and $P(x)$ hit the points.

$R(x) = Q(x) - P(x)$ has $d+1$ roots and is degree d.
Contradiction. $\square$

Must prove **Roots fact.**

Polynomial Division.

Divide $4x^2 - 3x + 2$ by $(x-3)$ modulo 5.

```
                 4 x + 4 r 4
           -----------------
     x - 3 ) 4x^2 - 3 x + 2
             4x^2 - 2x
             ----------
                    4x + 2
                    4x - 2
                    -------
                         4
```

$4x^2 - 3x + 2 \equiv (x-3)(4x+4) + 4 \pmod 5$

In general, divide $P(x)$ by $(x-a)$ gives $Q(x)$ and remainder r.

That is, $P(x) = (x-a)Q(x) + r$

Only d roots.

Lemma 1: $P(x)$ has root a iff $P(x)/(x-a)$ has remainder 0:
$P(x) = (x-a)Q(x)$.

Proof: $P(x) = (x-a)Q(x) + r$.
Plugin a: $P(a) = r$.
It is a root if and only if $r = 0$. $\square$

Lemma 2: $P(x)$ has d roots; $r_1, \ldots, r_d$ then
$P(x) = c(x-r_1)(x-r_2)\cdots(x-r_d)$.
Proof Sketch: By induction.

Induction Step: $P(x) = (x-r_1)Q(x)$ by Lemma 1. $Q(x)$ has smaller degree so use the induction hypothesis. $\square$

$d+1$ roots implies degree is at least $d+1$.

Roots fact: Any degree d polynomial has at most d roots.

Finite Fields

Proof works for reals, rationals, and complex numbers.

..but not for integers, since no multiplicative inverses.

Arithmetic modulo a prime p has multiplicative inverses..

..and has only a finite number of elements.

Good for computer science.

Arithmetic modulo a prime m is a **finite field** denoted by F_m or $GF(m)$.

Intuitively, a field is a set with operations corresponding to addition, multiplication, and division.