Today.

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Polynomials.

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Secret Sharing.

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Correcting for loss or even corruption.

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Share secret among *n* people.

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Two points make a line.

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Share secret among n people.

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Efficient: minimize storage.

The idea of the day.

Two points make a line.
Lots of lines go through one point.

Polynomials

A **polynomial**

$$P(x) = a_d x^d + a_{d-1} x^{d-1} \cdots + a_0.$$

is specified by **coefficients** $a_d, \ldots a_0$.

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Polynomials $P(x)$ with arithmetic modulo p: [1] $a_i \in \{0, \ldots, p-1\}$ and

$$P(x) = a_d x^d + a_{d-1} x^{d-1} \cdots + a_0 \pmod{p},$$

for $x \in \{0, \ldots, p-1\}$.

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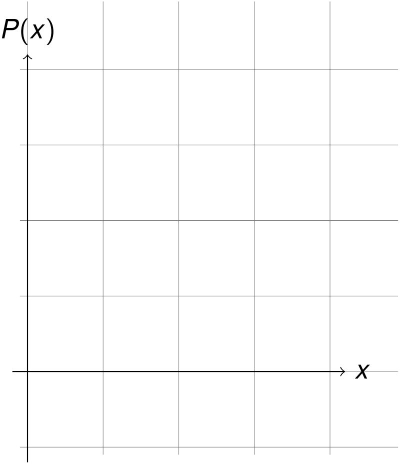

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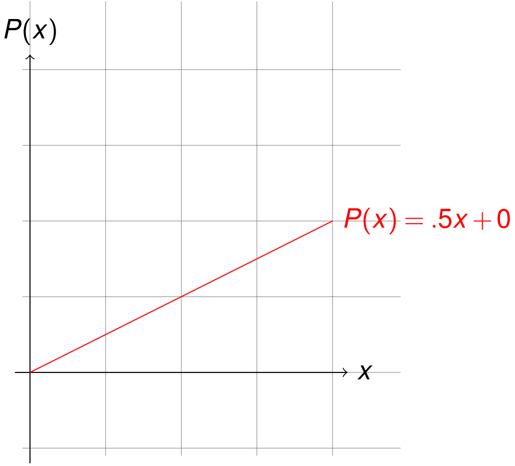

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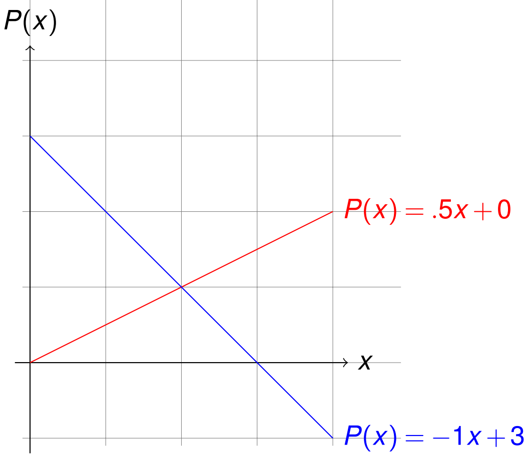

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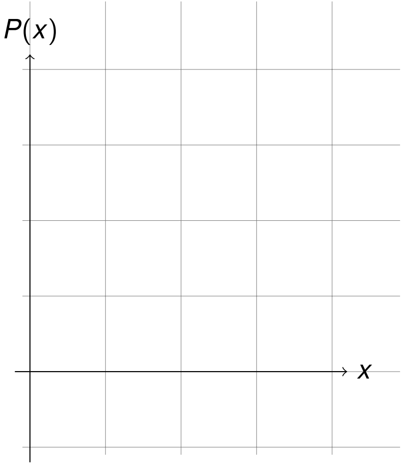

Parabola: $P(x) = a_2 x^2 + a_1 x + a_0$

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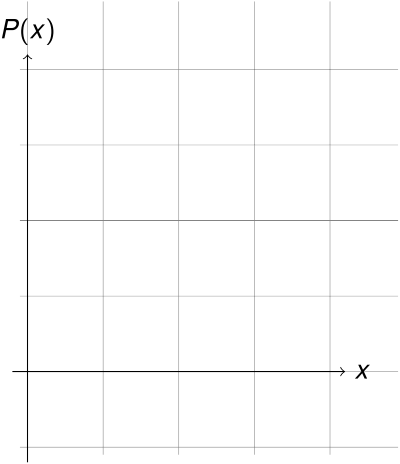

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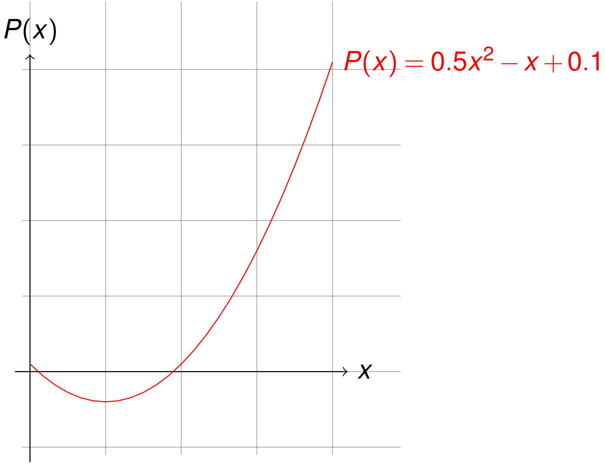

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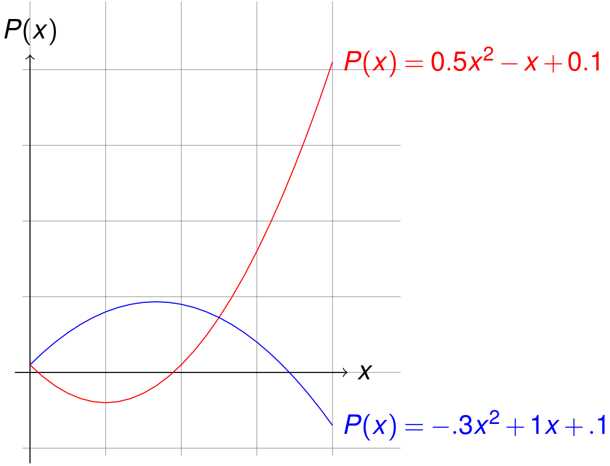

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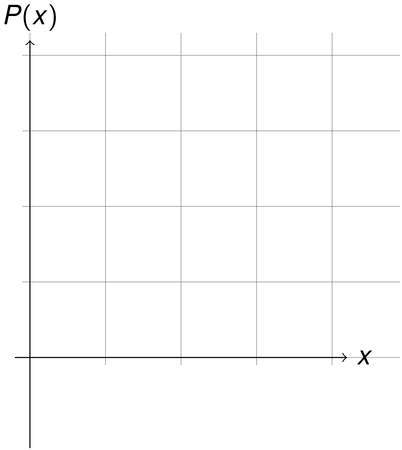

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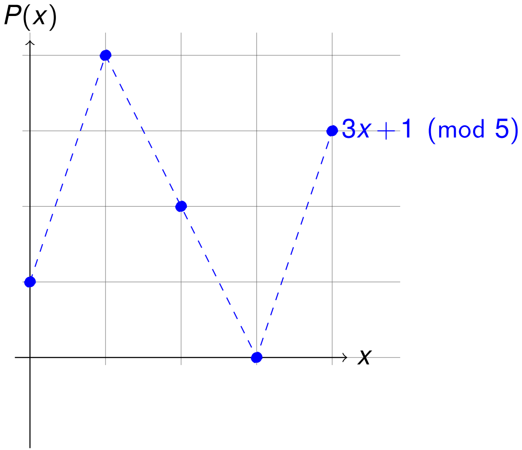

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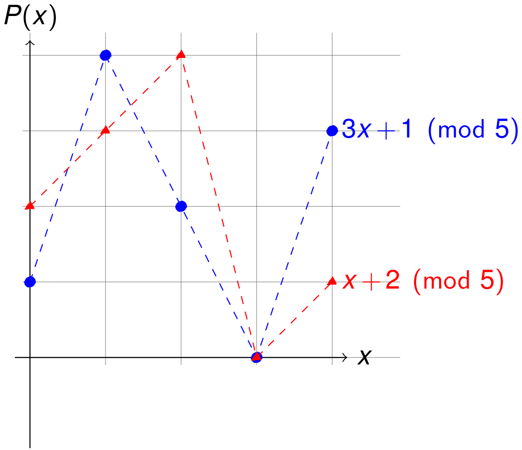

Finding an intersection.
$x+2 \equiv 3x+1 \pmod 5$
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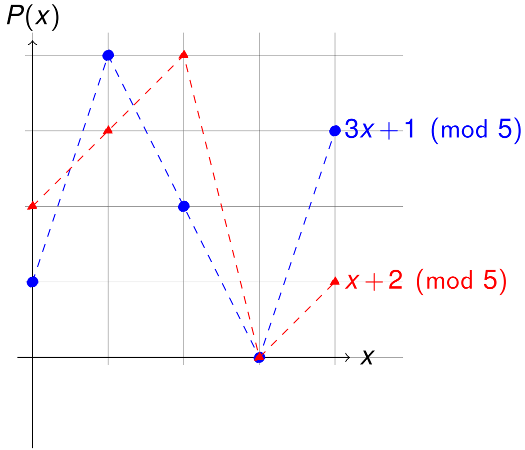

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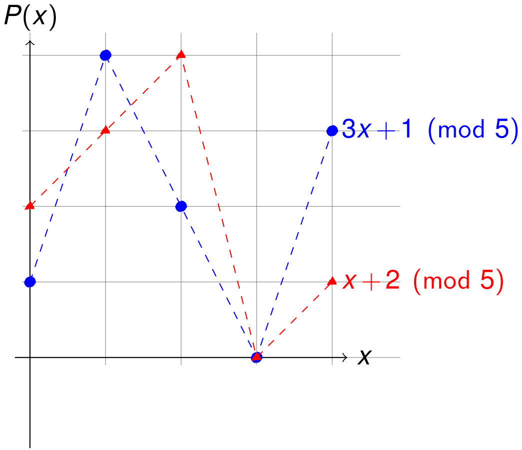

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Good when modulus is prime!!

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Fact: Exactly 1 degree $\leq d$ polynomial contains $d + 1$ points. [2]

[2] Points with different x values.

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Modular Arithmetic Fact: Exactly 1 degree $\leq d$ polynomial with arithmetic modulo prime p contains $d+1$ pts.

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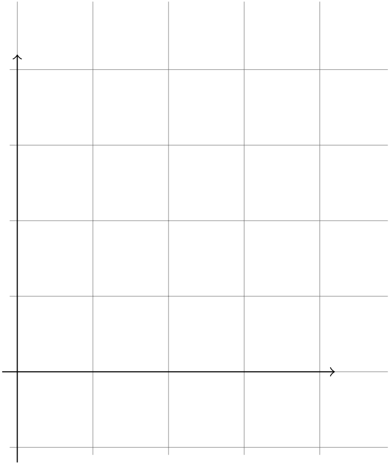
3 points determine a parabola.



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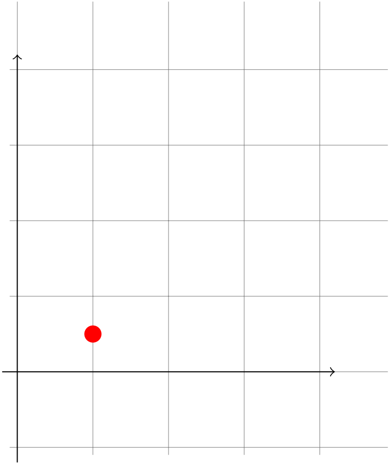

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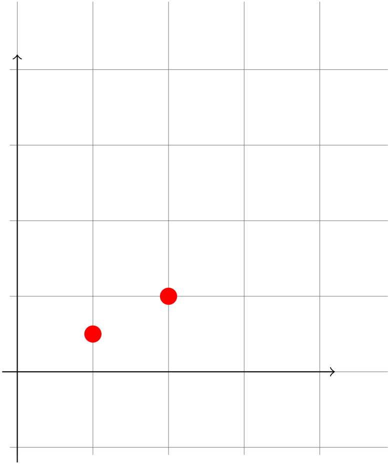

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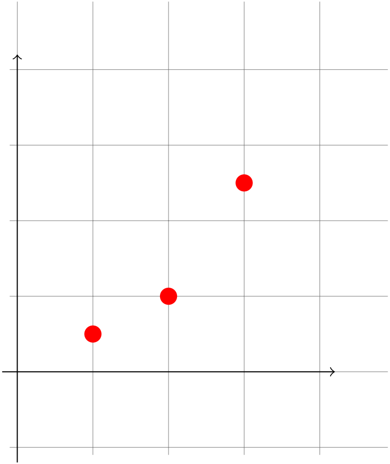

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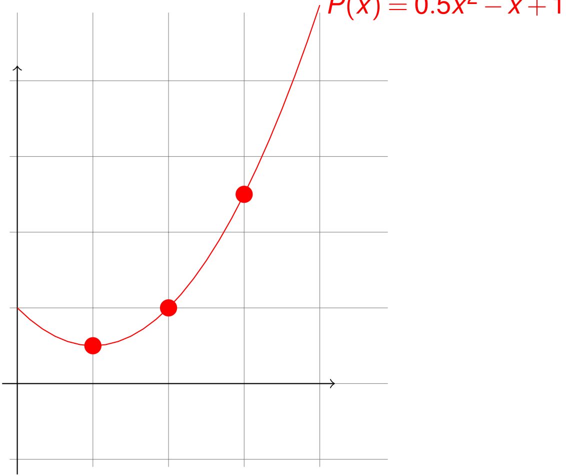

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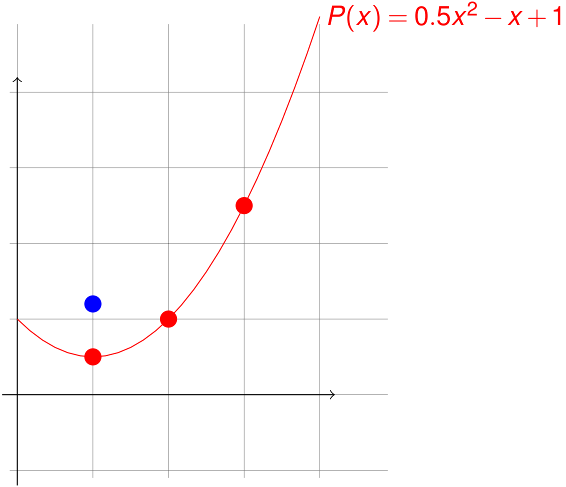

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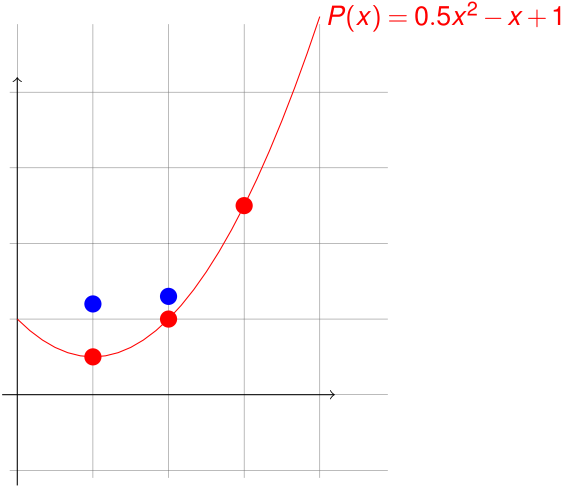

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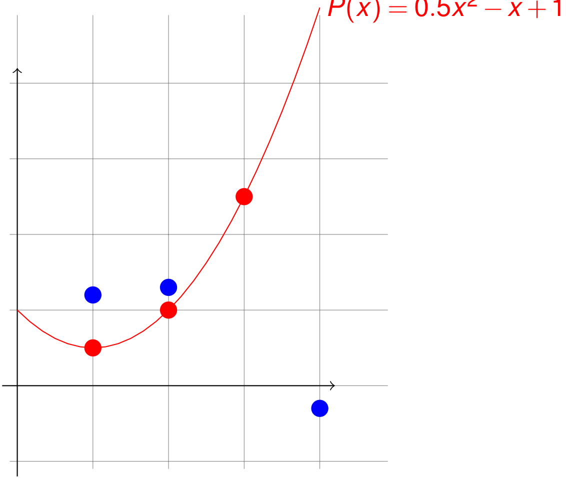

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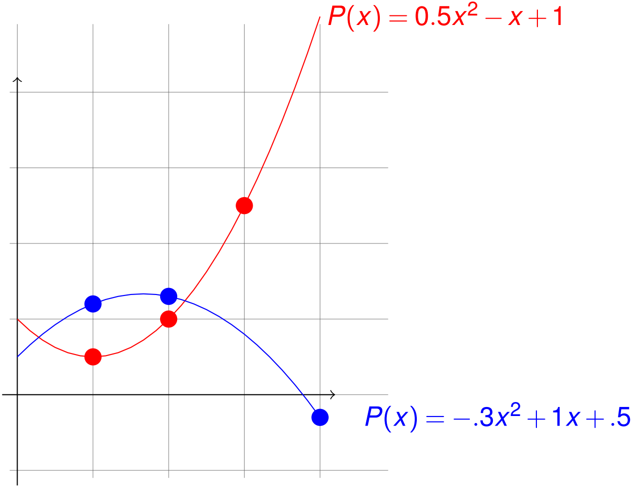

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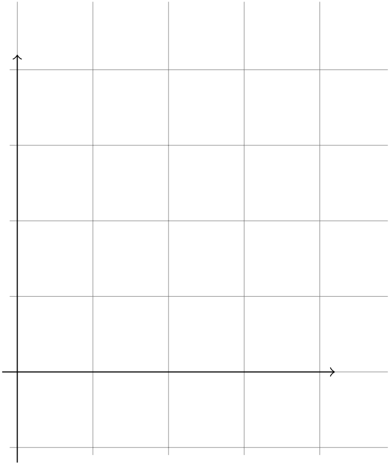

There is $P(x)$ contains blue points and *any* $(0, y)$!

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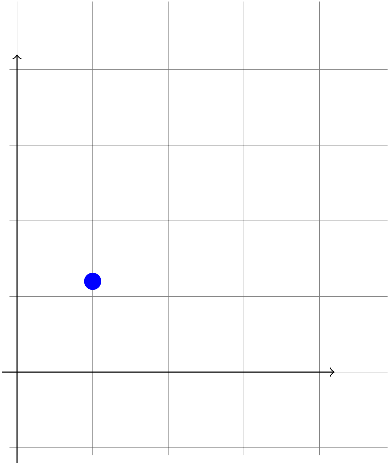

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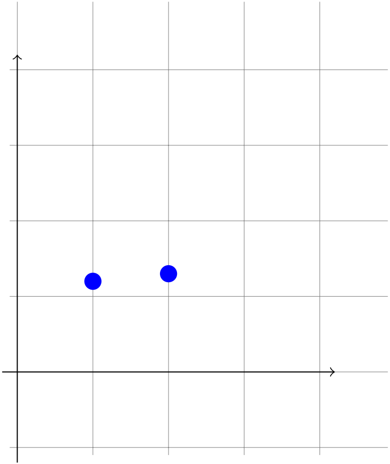

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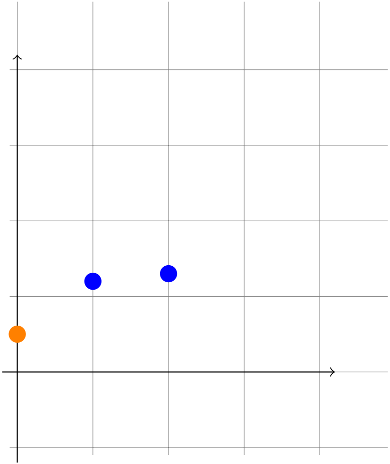

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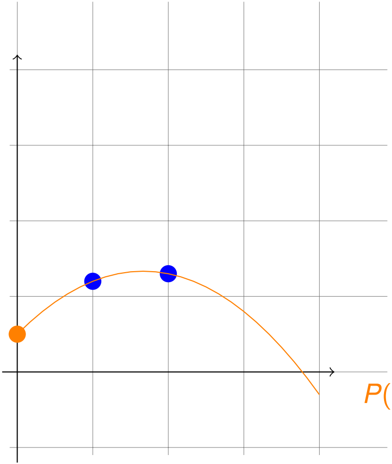

$P(x) = -.3x^2 + 1x + .5$

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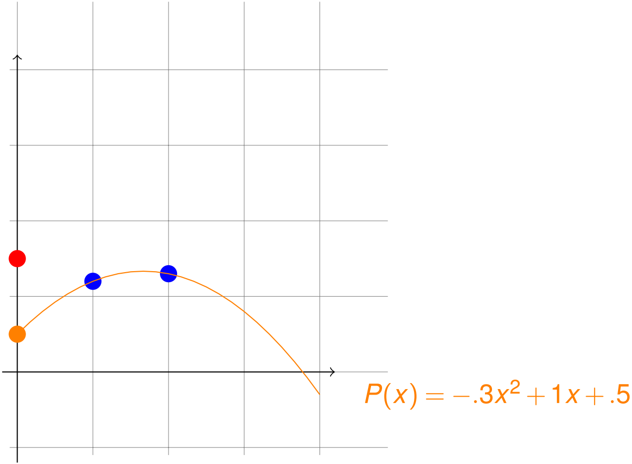

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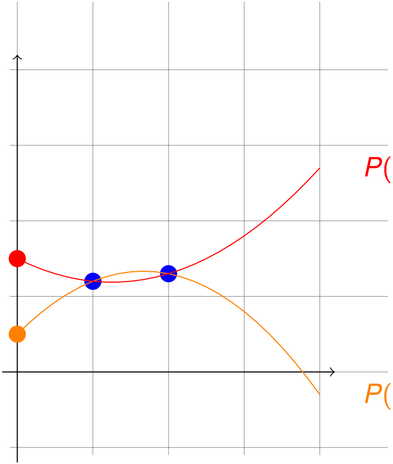

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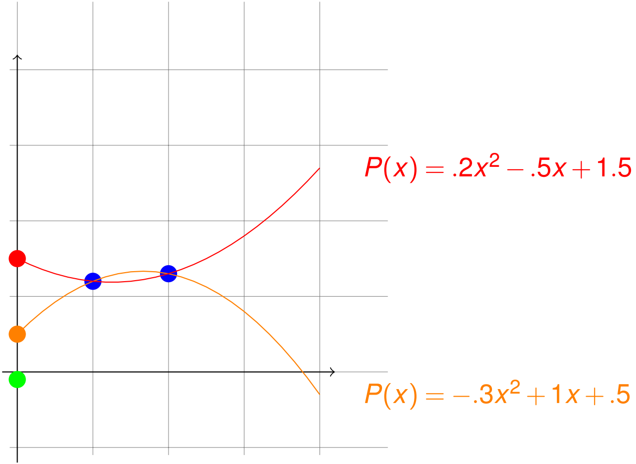

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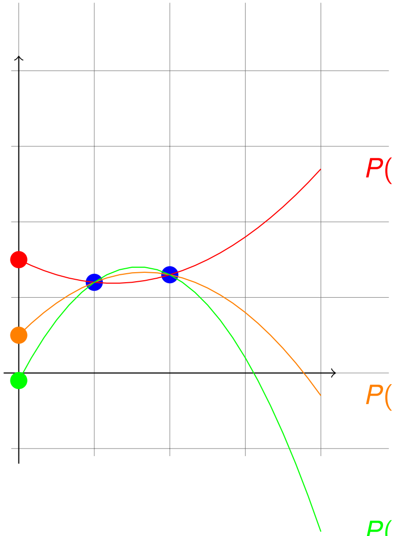

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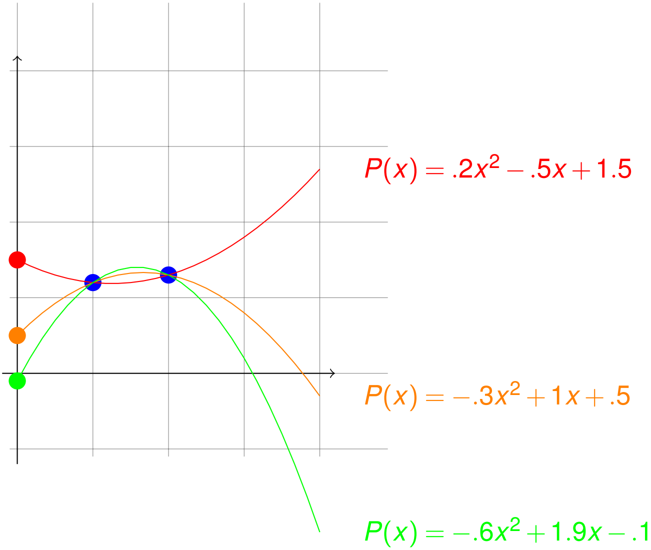

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Knowing k pts $\implies$ only one $P(x)$ $\implies$ evaluate $P(0)$.
Secrecy: Any $k-1$ shares give nothing.

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Multiplicative inverses due to $gcd(x,p) = 1$, forall $x \in \{1,\ldots,p-1\}$

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Construction proves the existence of the polynomial!

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In general, divide $P(x)$ by $(x-a)$ gives $Q(x)$ and remainder r.

That is, $P(x) = (x-a)Q(x) + r$

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Intuitively, a field is a set with operations corresponding to addition, multiplication, and division.