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Sample with replacement and order doesn't matter: $\binom{k+n-1}{n-1}$.

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Dividing 5 dollars among Alice, Bob and Eve.

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Two indistinguishable jokers in 54 card deck.
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Choosing a subset of size k is same

Combinatorial Proofs.

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$\square$

Pascal’s Triangle

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	0	
1		1

Pascal's Triangle

$$
\begin{array}{ccccc}
 & & 0 & & \\
 & 1 & & 1 & \\
1 & & 2 & & 1
\end{array}
$$

Pascal's Triangle

			0			
		1		1		
	1		2		1	
1		3		3		1

Pascal's Triangle

				0				
			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1

Pascal's Triangle

0
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Pascal's Triangle

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.

Pascal's Triangle

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
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\end{array}
$$

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Foil (4 terms)

Pascal's Triangle

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.

Foil (4 terms) on steroids:

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$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
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\end{array}
$$

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2^n terms:

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$$\begin{array}{ccccccccc} &&&& 0 &&&& \\ &&& 1 && 1 &&& \\ && 1 && 2 && 1 && \\ & 1 && 3 && 3 && 1 & \\ 1 && 4 && 6 && 4 && 1 \end{array}$$

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Foil (4 terms) on steroids:
2^n terms: choose 1 or x from each term $(1+x)$.

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$$
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 & & & & 0 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
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Simplify: collect all terms corresponding to x^k.

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  Coefficient of x^k $\binom{n}{k}$: choose k terms with x in product.

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$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
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$$
\begin{array}{ccc}
 & \binom{0}{0} & \\
\binom{1}{0} & & \binom{1}{1}
\end{array}
$$

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Pascal's Triangle

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Pascal's rule $\Longrightarrow$ $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$?

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How many size k subsets of $n+1$?
How many contain the first element?

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How many size k subsets of $n+1$?
How many contain the first element?
Chose first element, need $k-1$ more from remaining n elements.

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Without and with first element $\rightarrow$ disjoint.

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So, $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$. $\square$

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Must choose $k-1$ elements from $n-i$ remaining elements.

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Add them up to get the total number of subsets of size k which is also $\binom{n+1}{k}$.

$\square$

Binomial Theorem: $x = 1$

Theorem: $2^n = \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{0}$

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element i **is in** or **is not** in the subset: 2 poss.

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Simple Inclusion/Exclusion

Sum Rule: For disjoint sets S and T, $|S \cup T| = |S| + |T|$
Used to reason about all subsets
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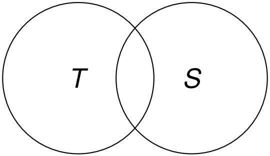

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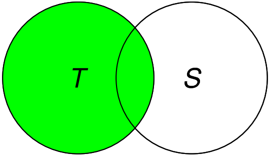

In T. $\implies$ $|T|$

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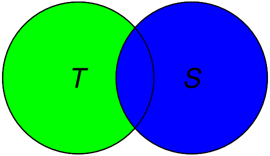

In T. $\implies |T|$
In S. $\implies + \ |S|$

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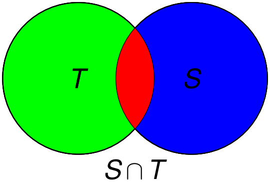

In T. $\implies |T|$
In S. $\implies + \ |S|$
Elements in $S \cap T$ are counted twice.

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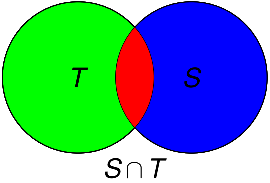

In T. $\implies$ $|T|$
In S. $\implies$ $+ \ |S|$
Elements in $S \cap T$ are counted twice.
Subtract. $\implies$ $-|S \cap T|$

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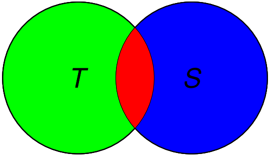

$S \cap T$

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In S. $\implies + \ |S|$
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$$|S \cup T| = |S| + |T| - |S \cap T|$$

Using Inclusion/Exclusion.

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Example: How many 10-digit phone numbers have 7 as their first or second digit?

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Example: How many 10-digit phone numbers have 7 as their first or second digit?

S = phone numbers with 7 as first digit.$|S| = 10^9$

Using Inclusion/Exclusion.

Inclusion/Exclusion Rule: For any S and T,
$|S \cup T| = |S| + |T| - |S \cap T|$.

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$S \cap T$ = phone numbers with 7 as first and second digit.

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$S \cap T$ = phone numbers with 7 as first and second digit. $|S \cap T| = 10^8$.

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T = phone numbers with 7 as second digit. $|T| = 10^9$.

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Answer: $|S| + |T| - |S \cap T| = 10^9 + 10^9 - 10^8$.

Summary.

First Rule of counting:

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First Rule of counting: Objects from a sequence of choices:

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n_i possibilitities for ith choice :

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First Rule of counting: Objects from a sequence of choices:
n_i possibilititities for ith choice : $n_1 \times n_2 \times \cdots \times n_k$ objects.

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n_i possibilitities for ith choice : $n_1 \times n_2 \times \cdots \times n_k$ objects.

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n_i possibilitities for ith choice : $n_1 \times n_2 \times \cdots \times n_k$ objects.

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Count with order:

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Add number of each subtract intersection of sets.

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LHS: $\binom{n}{k-1}$ counts subsets of $n+1$ items with first item.

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Disjoint

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Disjoint – so add!

Midterm Review

Now...

First there was logic...

A statement is true or false.

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A statement is true or false.

Statements?

First there was logic...

A statement is true or false.

Statements?

$3 = 4 - 1$?

First there was logic...

A statement is true or false.

Statements?

$3 = 4 - 1$? Statement!

First there was logic...

A statement is true or false.

Statements?

$3 = 4 - 1$? Statement!

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Statements?

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Statements?

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3 ? Not a statement!

First there was logic...

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Statements?

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$n = 3$?

First there was logic...

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Statements?

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$n = 3$? Not a statement...

First there was logic...

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Statements?

$3 = 4 - 1$? Statement!

$3 = 5$? Statement!

3 ? Not a statement!

$n = 3$? Not a statement...but a predicate.

First there was logic...

A statement is true or false.

Statements?

$3 = 4 - 1$? Statement!

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$n = 3$? Not a statement...but a predicate.

Predicate: Statement with free variable(s).

First there was logic...

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$3 = 4 - 1$? Statement!

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Predicate: Statement with free variable(s).

Example: $x = 3$

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Example: $x = 3$

Given a value for x, becomes a statement.

Predicate?

$n > 3$?

First there was logic...

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Statements?

$3 = 4 - 1$? Statement!

$3 = 5$? Statement!

3 ? Not a statement!

$n = 3$? Not a statement...but a predicate.

Predicate: Statement with free variable(s).

Example: $x = 3$

Given a value for x, becomes a statement.

Predicate?

$n > 3$? Predicate: $P(n)$!

First there was logic...

A statement is true or false.

Statements?

$3 = 4 - 1$? Statement!

$3 = 5$? Statement!

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Predicate: Statement with free variable(s).

Example: $x = 3$

Given a value for x, becomes a statement.

Predicate?

$n > 3$? Predicate: $P(n)$!

$x = y$?

First there was logic...

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$3 = 4 - 1$? Statement!

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Predicate: Statement with free variable(s).

Example: $x = 3$

Given a value for x, becomes a statement.

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$n > 3$? Predicate: $P(n)$!

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Statements?

- $3 = 4 - 1$? Statement!
- $3 = 5$? Statement!
- 3 ? Not a statement!
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Predicate: Statement with free variable(s).

- Example: $x = 3$
  - Given a value for x, becomes a statement.

Predicate?

- $n > 3$? Predicate: $P(n)$!
- $x = y$? Predicate: $P(x, y)$!
- $x + y$?

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$x + y$? No.

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$x + y$? No. An expression, not a statement.

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Quantifiers:

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Quantifiers:

$(\forall x)\ P(x)$. For every x, $P(x)$ is true.

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Statements?

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Quantifiers:

$(\forall x)\ P(x)$. For every x, $P(x)$ is true.

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$(\forall n \in N), n^2 \geq n$.

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$(\forall x \in R)(\exists y \in R) y > x$.

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Connecting Statements

$A \wedge B$, $A \vee B$, $\neg A$.

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$A \wedge B$, $A \vee B$, $\neg A$.

You got this!

Connecting Statements

$A \wedge B$, $A \vee B$, $\neg A$.

You got this!

Propositional Expressions and Logical Equivalence

Connecting Statements

$A \land B$, $A \lor B$, $\neg A$.

You got this!

Propositional Expressions and Logical Equivalence

$(A \implies B) \equiv (\neg A \lor B)$

Connecting Statements

$A \wedge B$, $A \vee B$, $\neg A$.

You got this!

Propositional Expressions and Logical Equivalence

$(A \implies B) \equiv (\neg A \vee B)$
$\neg(A \vee B) \equiv (\neg A \wedge \neg B)$

Connecting Statements

$A \wedge B$, $A \vee B$, $\neg A$.

You got this!

Propositional Expressions and Logical Equivalence

$(A \implies B) \equiv (\neg A \vee B)$
$\neg(A \vee B) \equiv (\neg A \wedge \neg B)$

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Propositional Expressions and Logical Equivalence

$(A \implies B) \equiv (\neg A \vee B)$
$\neg(A \vee B) \equiv (\neg A \wedge \neg B)$

Proofs: truth table or manipulation of known formulas.

Connecting Statements

$A \wedge B$, $A \vee B$, $\neg A$.

You got this!

Propositional Expressions and Logical Equivalence

$(A \implies B) \equiv (\neg A \vee B)$
$\neg(A \vee B) \equiv (\neg A \wedge \neg B)$

Proofs: truth table or manipulation of known formulas.

$(\forall x)(P(x) \wedge Q(x)) \equiv (\forall x)P(x) \wedge (\forall x)Q(x)$

..and then proofs...

Direct: $P \implies Q$

..and then proofs...

Direct: $P \implies Q$

Example: a is even $\implies a^2$ is even.

..and then proofs...

Direct: $P \implies Q$

Example: a is even $\implies a^2$ is even.

Approach: What is even?

..and then proofs...

Direct: $P \implies Q$

Example: a is even $\implies a^2$ is even.

Approach: What is even? $a = 2k$

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Contrapositive: $P \implies Q$

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Useful for prove something does not exist:

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Useful for prove something does not exist:

Example: rational representation of $\sqrt{2}$ does not exist.

Example: finite set of primes does not exist.

Example: rogue couple does not exist.

...jumping forward..

Contradiction in induction:

...jumping forward..

Contradiction in induction:

contradict place where induction step doesn't hold.

...jumping forward..

Contradiction in induction:

contradict place where induction step doesn't hold.

Well Ordering Principle.

...jumping forward..

Contradiction in induction:
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Well Ordering Principle.
 Stable Marriage:

...jumping forward..

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Well Ordering Principle.
 Stable Marriage:
  first day where canditate gets worse job on string.

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first day where canditate gets worse job on string.

first day where any job is rejected by optimal candidate.

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  Stable Marriage:
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$$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\ P(n).$$

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Thm: For all $n \geq 1$, $8 | 3^{2n} - 1$.

...and then induction...

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Thm: For all $n \geq 1$, $8 | 3^{2n} - 1$.

Induction on n.

...and then induction...

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Induction on n.

Base: $8|3^2-1$.

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$3^{2n+2} - 1 =$

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Induction on n.

Base: $8 | 3^2 - 1$.

Induction Hypothesis: Assume $P(n)$: True for some n.

Induction Step: Prove $P(n+1)$

$$3^{2n+2} - 1 = 9(3^{2n}) - 1$$

...and then induction...

$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\ P(n).$

Thm: For all $n \geq 1$, $8|3^{2n}-1$.

Induction on n.

Base: $8|3^2-1$.

Induction Hypothesis: Assume $P(n)$: True for some n.
$(3^{2n}-1 = 8d)$

Induction Step: Prove $P(n+1)$

$3^{2n+2}-1 = 9(3^{2n})-1$ (by induction hypothesis)

...and then induction...

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Induction Step: Prove $P(n+1)$

$$\begin{aligned} 3^{2n+2}-1 &= 9(3^{2n})-1 \quad \text{(by induction hypothesis)} \\ &= 9(8d+1)-1 \end{aligned}$$

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Thm: For all $n \geq 1$, $8 | 3^{2n} - 1$.

Induction on n.

Base: $8 | 3^2 - 1$.

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$(3^{2n} - 1 = 8d)$

Induction Step: Prove $P(n+1)$

$$\begin{aligned} 3^{2n+2} - 1 &= 9(3^{2n}) - 1 \quad \text{(by induction hypothesis)} \\ &= 9(8d+1) - 1 \\ &= 72d + 8 \end{aligned}$$

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$$
\begin{aligned}
3^{2n+2} - 1 &= 9(3^{2n}) - 1 \quad \text{(by induction hypothesis)} \\
&= 9(8d+1) - 1 \\
&= 72d + 8 \\
&= 8(9d+1)
\end{aligned}
$$

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$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\ P(n)$.

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Induction on n.

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Divisible by 8.

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Divisible by 8.

□

Stable Matching: a study in definitions and WOP.

n-jobs, *n*-candidate.

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Each entity has completely ordered preference list

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Pairing.
Set of pairs (m_i, w_j) containing all entities *exactly* once.

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How many pairs?

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Set of pairs (m_i, w_j) containing all entities *exactly* once.
How many pairs? n.

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Entities in pair are **partners** in pairing.

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Rogue Couple in a pairing.
A m_j and w_k who like each other more than their partners

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Stable Pairing.
Pairing with no rogue couples.

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Pairing.
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Stable Pairing.
Pairing with no rogue couples.

Does stable pairing exist?

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Stable Pairing.
Pairing with no rogue couples.

Does stable pairing exist?

No, for roommates problem.

TMA.

Job Propose or reject Matching Algorithm:

TMA.

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Each Day:

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All jobs propose to favorite non-rejecting candidate.

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Useful Algorithmic Definitions:

TMA.

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"Propose and Reject."

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"Propose and Reject." : Either jobs propose or candidate.

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"Propose and Reject." : Either jobs propose or candidate. But not both.
Traditional propose and reject where jobs propose.

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Key Property: Improvement Lemma:

TMA.

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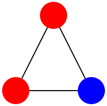

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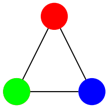

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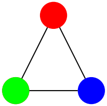

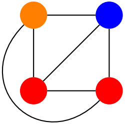

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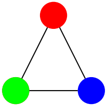

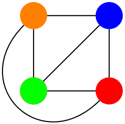

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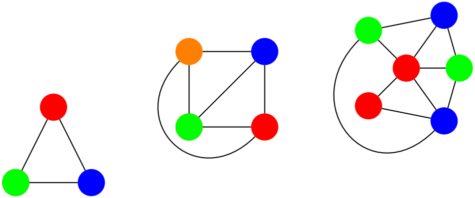

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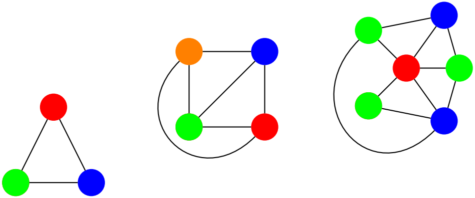

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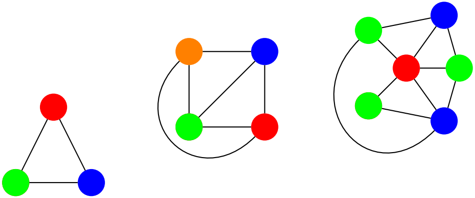

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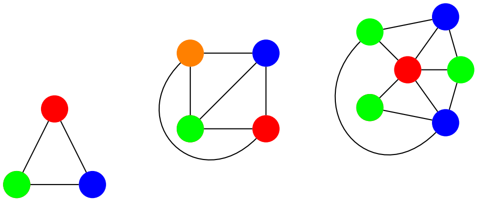

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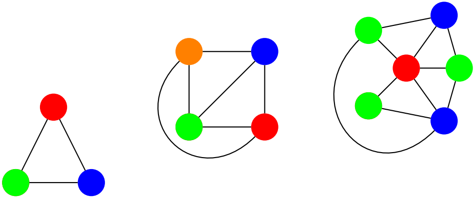

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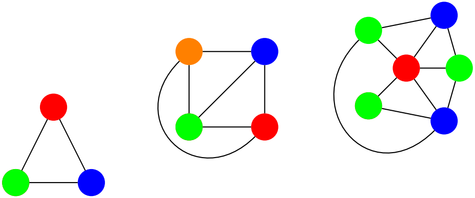

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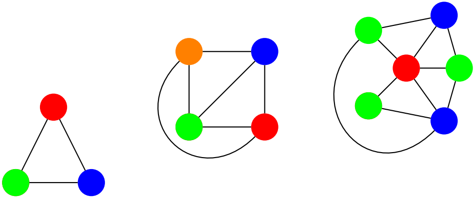

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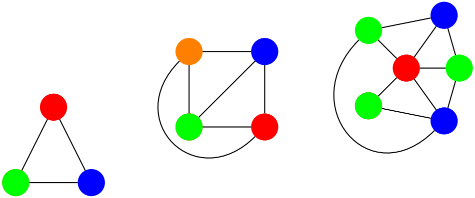

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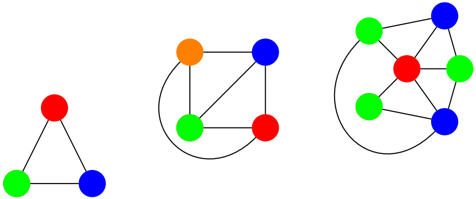

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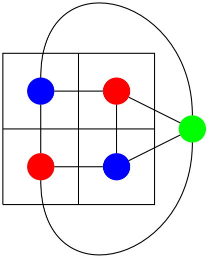

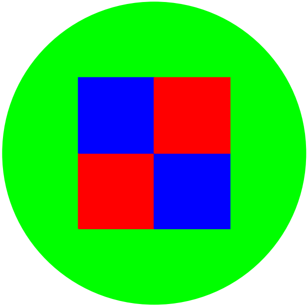

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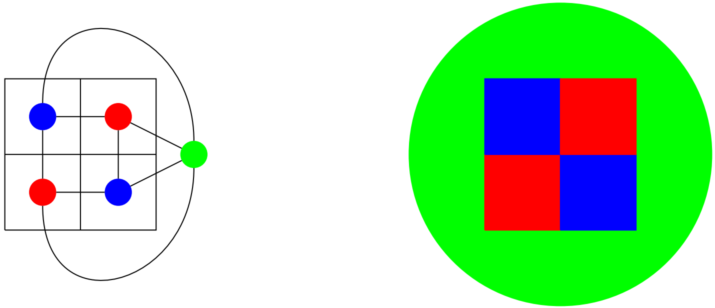

Four color theorem is about planar graphs!

Six color theorem.

Theorem: Every planar graph can be colored with six colors.

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Five color theorem: summary.

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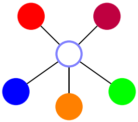

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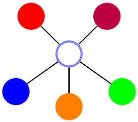

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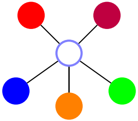

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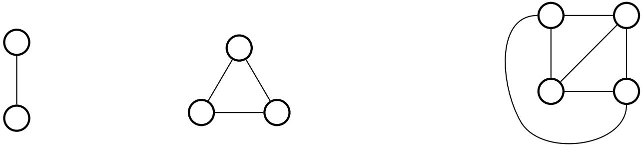
One will work.

Graph Types: Complete Graph.



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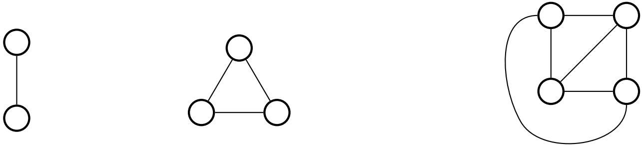

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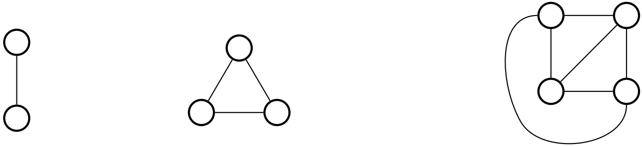

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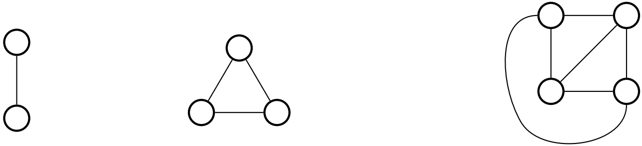

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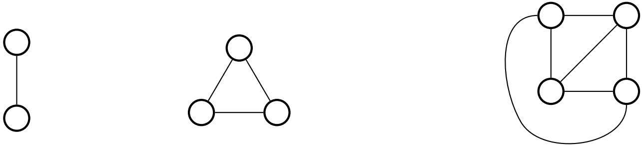

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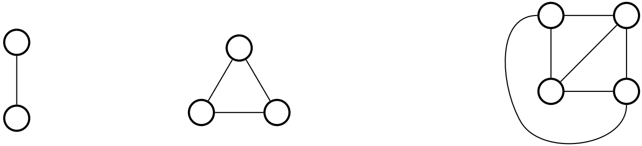

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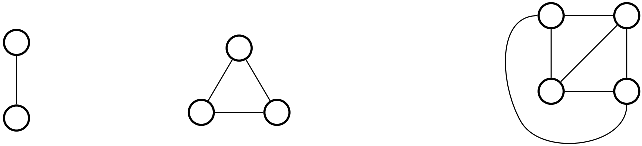

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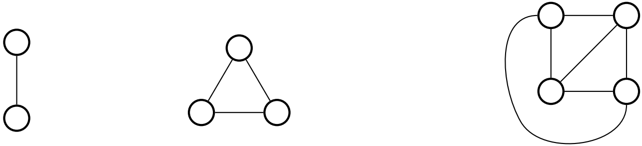

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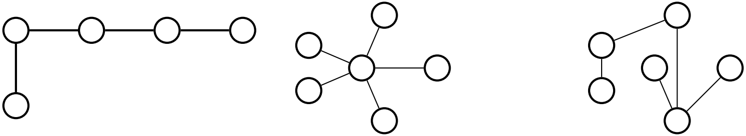
Very connected.
Lots of edges: $n(n-1)/2$.

Trees.



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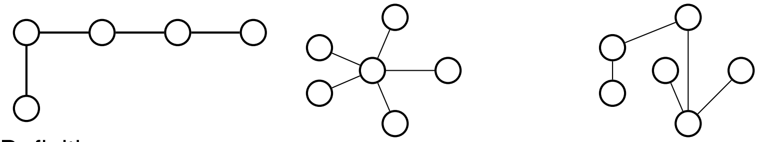

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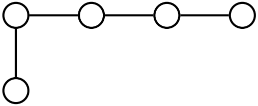
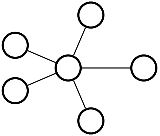
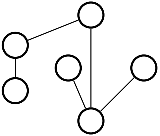

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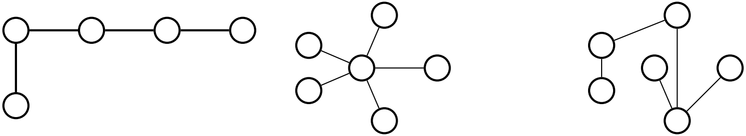

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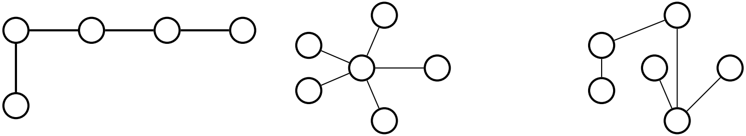

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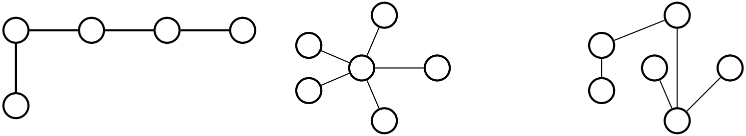

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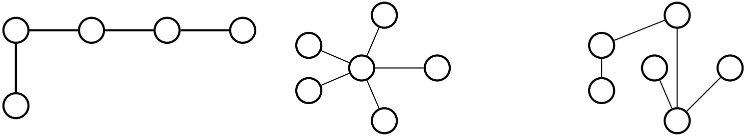

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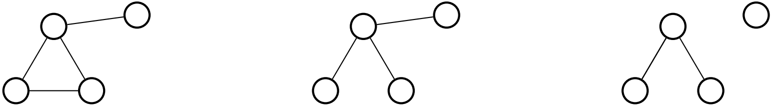

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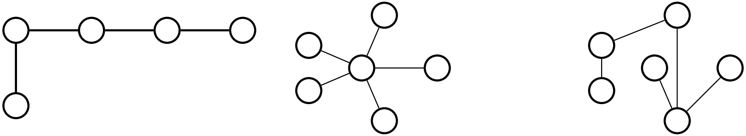

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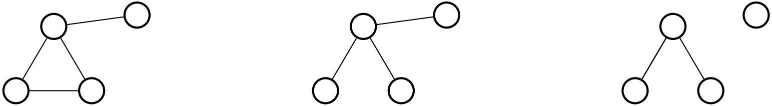

Minimally connected, minimum number of edges to connect.

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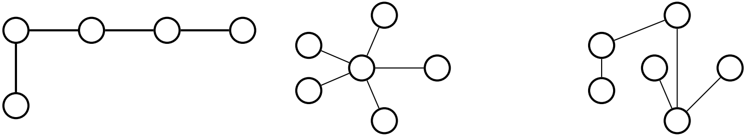

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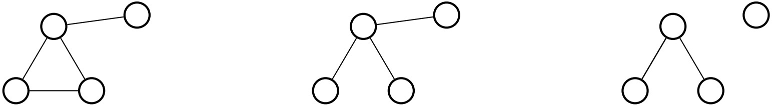

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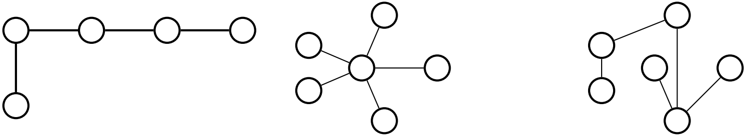

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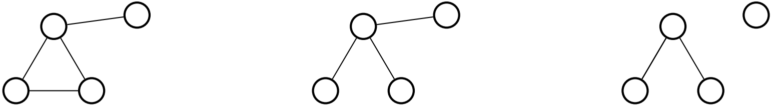

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Property:
Can remove a single node and break into components of size at most $|V|/2$.

Hypercube

Hypercubes.

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Hypercubes. Really connected.

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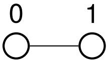

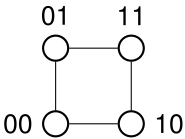

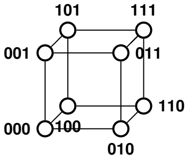

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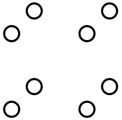

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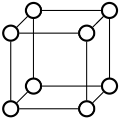

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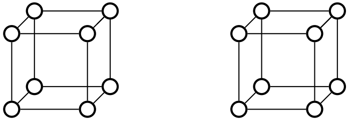

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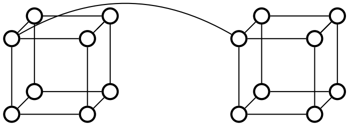

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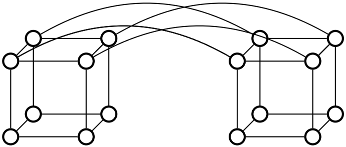

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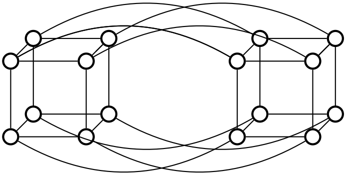

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FYI:

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FYI: Also cuts represent boolean functions.

Hypercube:properties

Rudrata Cycle: cycle that visits every node.
Eulerian? If n is even.

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Good communication network!

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Arithmetic modulo m.
  Elements of equivalence classes of integers.

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Additive inverses are intuitively negative numbers.

Modular Arithmetic and multiplicative inverses.

$3^{-1} \pmod 7$?

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x has inverse modulo m if and only if $gcd(x, m) = 1$.

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Hand calculation: egcd.

Extended GCD: egcd$(7, 60) = 1$.

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$$7(0) + 60(1) \quad = \quad 60$$

Hand calculation: egcd.

Extended GCD: $\text{egcd}(7,60) = 1$.
egcd(7,60).

$$
\begin{aligned}
7(0) + 60(1) &= 60 \\
7(1) + 60(0) &= 7
\end{aligned}
$$

Hand calculation: egcd.

Extended GCD: $\text{egcd}(7,60) = 1$.
egcd(7,60).

$$
\begin{aligned}
7(0)+60(1) &= 60 \\
7(1)+60(0) &= 7 \\
7(-8)+60(1) &= 4
\end{aligned}
$$

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$$\begin{aligned} 7(0) + 60(1) &= 60 \\ 7(1) + 60(0) &= 7 \\ 7(-8) + 60(1) &= 4 \\ 7(9) + 60(-1) &= 3 \end{aligned}$$

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$$
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Confirm: $-119 + 120 = 1$

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Confirm: $-119 + 120 = 1$

$d = e^{-1} = -17 = 43 = \pmod{60}$

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e with $\gcd(e, (p-1)(q-1)) = 1$.

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RSA, Public Key, and Signatures.

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Announce $(C, S(C))$

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$E(D(C,k),K) = (C^d)^e = C \pmod{N}$

Fermat/RSA

$3^6 \pmod 7$?

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$2^14 \pmod{21}$?

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$2^{12} \pmod{21}$? 1.
$21 = (3)(7)$ $(p-1)(q-1) = (2)(6) = 12$
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$2^1 4 \pmod{21}$? 4.

Fermat/RSA

$3^6 \pmod 7$? 1. Fermat: $p = 7$, $p - 1 = 6$
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Lemma 1: $P(x)$ has root a iff $P(x)/(x-a)$ has remainder 0:
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Roots fact: Any degree $\leq d$ polynomial has at most d roots.

Secret Sharing

Modular Arithmetic Fact: Exactly one polynomial degree $\leq d$ over $GF(p)$, $P(x)$, that hits $d+1$ points.

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Within 1 of optimal number of bits.

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Communicate n packets, with k erasures.

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How many packets?

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Polynomial division! $P(x) = Q(x)/E(x)$!

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Reconstruct error polynomial, $E(X)$, and $P(x)$!
Nonlinear equations.
Reconstruct $E(x)$ and $Q(x) = E(x)P(x)$. Linear Equations.
Polynomial division! $P(x) = Q(x)/E(x)$!

Reed-Solomon codes.

Summary. Error Correction.

Communicate n packets, with k erasures.

How many packets? $n+k$
How to encode? With polynomial, $P(x)$.
Of degree? $n-1$
Recover? Reconstruct $P(x)$ with any n points!

Communicate n packets, with k errors.

How many packets? $n+2k$
Why?
k changes to make diff. messages overlap
How to encode? With polynomial, $P(x)$. Of degree? $n-1$.
Recover?
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Reed-Solomon codes. Welsh-Berlekamp Decoding.

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Reed-Solomon codes. Welsh-Berlekamp Decoding. Perfection!

Midterm format

Time: 120 minutes.

Midterm format

Time: 120 minutes.

Some short answers.

Midterm format

Time: 120 minutes.

Some short answers.
 Get at ideas that you learned.

Midterm format

Time: 120 minutes.

Some short answers.
  Get at ideas that you learned.
   Know material well:

Midterm format

Time: 120 minutes.

Some short answers.
  Get at ideas that you learned.
   Know material well: fast,

Midterm format

Time: 120 minutes.

Some short answers.
Get at ideas that you learned.
Know material well: fast, correct.

Midterm format

Time: 120 minutes.

Some short answers.
Get at ideas that you learned.
Know material well: fast, correct.
Know material medium:

Midterm format

Time: 120 minutes.

Some short answers.
  Get at ideas that you learned.
   Know material well: fast, correct.
   Know material medium: slower,

Midterm format

Time: 120 minutes.

Some short answers.
 Get at ideas that you learned.
  Know material well: fast, correct.
  Know material medium: slower, less correct.

Midterm format

Time: 120 minutes.

Some short answers.

Get at ideas that you learned.

Know material well: fast, correct.

Know material medium: slower, less correct.

Know material not so well:

Midterm format

Time: 120 minutes.

Some short answers.
Get at ideas that you learned.

Know material well:	fast, correct.
Know material medium:	slower, less correct.
Know material not so well:	Uh oh.

Midterm format

Time: 120 minutes.

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Get at ideas that you learned.

Know material well:	fast, correct.
Know material medium:	slower, less correct.
Know material not so well:	Uh oh.

Some longer questions.

Midterm format

Time: 120 minutes.

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Some longer questions.
 Proofs,

Midterm format

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Some longer questions.
Proofs, algorithms,

Midterm format

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Midterm format

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  Proofs, algorithms, properties.
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Wrapup.

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Other issues....

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fa20@eecs70.org

Wrapup.

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  fa20@eecs70.org
  Private message on piazza.

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