Next up: how big is infinity.

Next up: how big is infinity.

- Countable
- Countably infinite.
- Enumeration

How big are the reals or the integers?

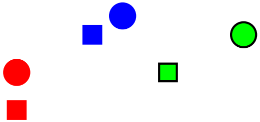
Infinite!

How big are the reals or the integers?

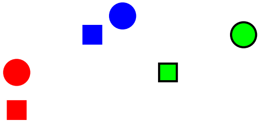
Infinite!

Is one bigger or smaller?

Same size?

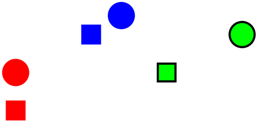


Same size?



Same number?

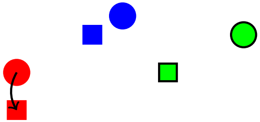
Same size?



Same number?

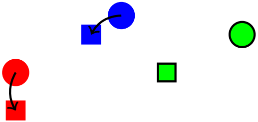
Make a function f : Circles $\to$ Squares.

Same size?



Same number?
Make a function f : Circles $\rightarrow$ Squares.
f(red circle) $=$ red square

Same size?



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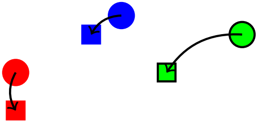
Same size?



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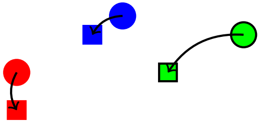
f(circle with black border) $=$ square with black border

Same size?



Same number?

Make a function f : Circles $\to$ Squares.

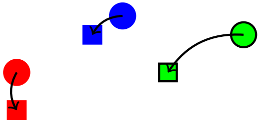
f(red circle) $=$ red square

f(blue circle) $=$ blue square

f(circle with black border) $=$ square with black border

One to one.

Same size?



Same number?

Make a function f : Circles $\rightarrow$ Squares.

f(red circle) $=$ red square

f(blue circle) $=$ blue square

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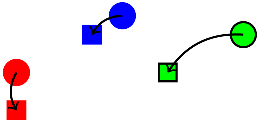
One to one. Each circle mapped to different square.

Same size?



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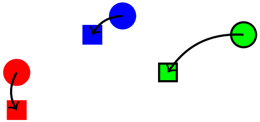
Same size?



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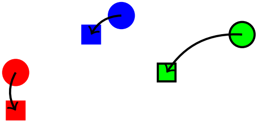
Onto.

Same size?



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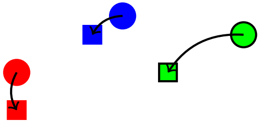
Onto. Each square mapped to from some circle .

Same size?



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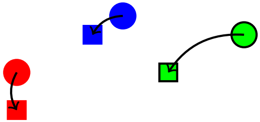
Onto: For all $s \in R$, $\exists c \in D, s = f(c)$.

Same size?



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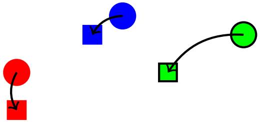

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Isomorphism principle: If there is $f : D \to R$ that is one to one and onto, then, $|D| = |R|$.

Isomorphism principle.

Given a function, $f : D \rightarrow R$.

Isomorphism principle.

Given a function, $f : D \to R$.

One to One:

Isomorphism principle.

Given a function, $f : D \rightarrow R$.
One to One:
For all $\forall x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.

Isomorphism principle.

Given a function, $f : D \to R$.
One to One:
For all $\forall x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.
or

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$\forall x, y \in D$, $f(x) = f(y) \implies x = y$.

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$f(\cdot)$ is a **bijection** if it is one to one and onto.

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Onto: For all $y \in R$, $\exists x \in D, y = f(x)$.

$f(\cdot)$ is a **bijection** if it is one to one and onto.

Isomorphism principle:
If there is a bijection $f : D \to R$ then $|D| = |R|$.

Countable.

How to count?

Countable.

How to count?

0,

Countable.

How to count?

0, 1,

Countable.

How to count?

0, 1, 2,

Countable.

How to count?

0, 1, 2, 3,

Countable.

How to count?

$0, 1, 2, 3, \ldots$

Countable.

How to count?

0, 1, 2, 3, $\ldots$

The Counting numbers.

Countable.

How to count?

0, 1, 2, 3, $\ldots$

The Counting numbers.
The natural numbers! N

Countable.

How to count?

0, 1, 2, 3, $\ldots$

The Counting numbers.
The natural numbers! N

Definition: S is **countable** if there is a bijection between S and some subset of N.

Countable.

How to count?

0, 1, 2, 3, ...

The Counting numbers.
The natural numbers! N

Definition: S is **countable** if there is a bijection between S and some subset of N.

If the subset of N is finite, S has finite **cardinality**.

Countable.

How to count?

0, 1, 2, 3, $\ldots$

The Counting numbers.
The natural numbers! N

Definition: S is **countable** if there is a bijection between S and some subset of N.

If the subset of N is finite, S has finite **cardinality**.

If the subset of N is infinite, S is **countably infinite**.

Where’s 0?

Which is bigger?

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

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Natural numbers. 0,

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Which is bigger?
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Natural numbers. $0, 1,$

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Where's 0?

More natural numbers!

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Where's 0?

More natural numbers!

Consider $f(z) = z - 1$.

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More natural numbers!

Consider $f(z) = z - 1$.

For any two $z_1 \neq z_2$

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For any natural number n, for $z = n + 1$, $f(z) = (n + 1) - 1$

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Onto for $\mathbb{N}$

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Bijection!

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Bijection! $\implies |\mathbb{Z}^+| = |\mathbb{N}|$.

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But.. but

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Bijection! $\implies |\mathbb{Z}^+| = |\mathbb{N}|$.

But.. but Where's zero? "Comes from 1."

A bijection is a bijection.

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Notice that there is a bijection between N and Z^+ as well.

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$f(n) = n + 1$.

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$f(n) = n + 1$. $0 \rightarrow 1,$

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Bijection from A to $B \implies$ a bijection from B to A.

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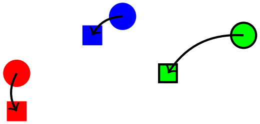

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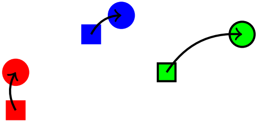

Inverse function!

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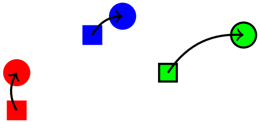


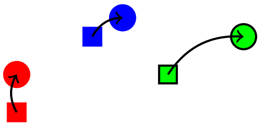
Inverse function!

Can prove equivalence either way.

A bijection is a bijection.

Notice that there is a bijection between N and Z^+ as well.
$f(n) = n+1$. $0 \to 1, 1 \to 2, \ldots$

Bijection from A to B $\implies$ a bijection from B to A.



Inverse function!

Can prove equivalence either way.
Bijection to or from natural numbers implies countably infinite.

More large sets.

E - Even natural numbers?

More large sets.

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$f : \mathbb{N} \to E$.

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Onto:

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$f : \mathbb{N} \rightarrow E$.

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Onto: $\forall e \in E$, $f(e/2) = e$.

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Evens are countably infinite.
Evens are same size as all natural numbers.

All integers?

What about Integers, Z?

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Define $f : N \to Z$.

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

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One-to-one: For $x \neq y$
if x is even and y is odd,
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$\ldots.$

Onto: For any $z \in Z$,
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if $z < 0$, $f(2|z| - 1) = z$ and $2|z| + 1 \in N$.

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Integers and naturals have same size!

Listings..

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1	-1

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2	1

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2	1
3	-2

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0	0
1	-1
2	1
3	-2
4	2

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Notice that: A listing “is” a bijection with a subset of natural numbers.

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If infinite: bijection with N.

Enumerability $\equiv$ countability.

Enumerating (listing) a set implies that it is countable.

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Any element x of S has *specific, finite* position in list.

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When do you get to -1?

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61A

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61A $\equiv$ streams!   Not Sp20/Fa20.

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All countably infinite sets have the same cardinality.

Enumeration example.

All binary strings.

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$B = \{0,1\}^*$.

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$B = \{\phi; , 0, 00, 000, 0000, ...\}$
Never get to 1.

More fractions?

Enumerate the rational numbers in order...

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$0, \ldots, 1/2, ..$

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Where is $1/2$ in list?

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After $1/3$, which is after $1/4$, which is after $1/5$...

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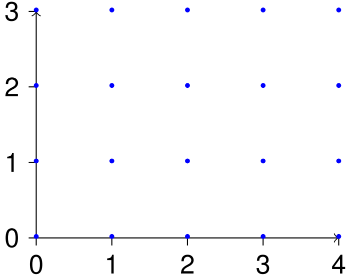

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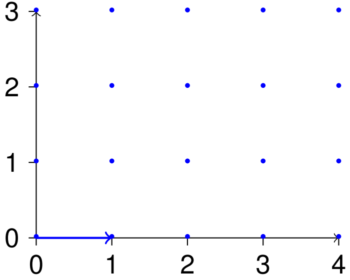

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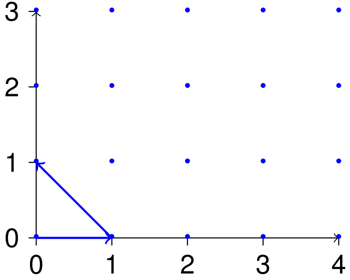

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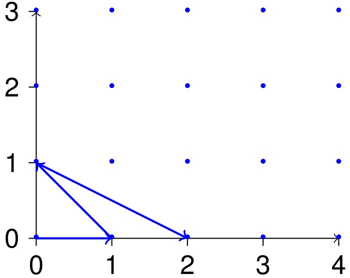

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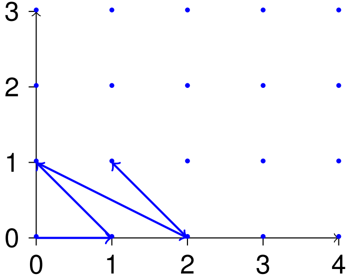

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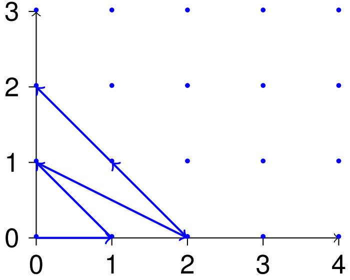

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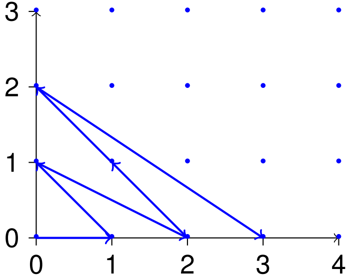

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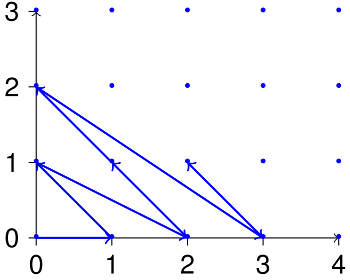

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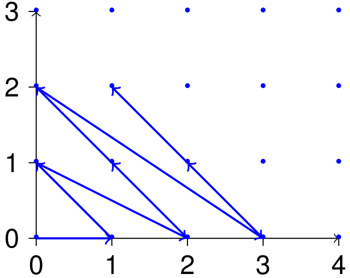

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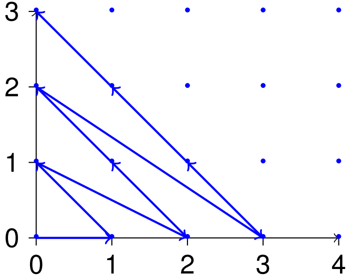

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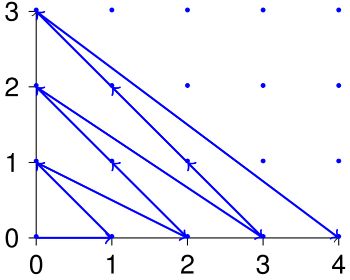

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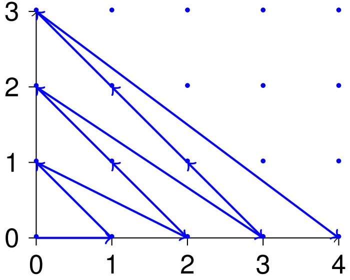

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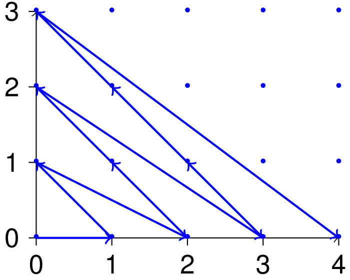

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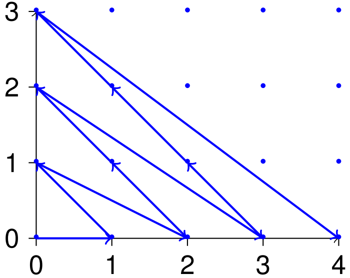

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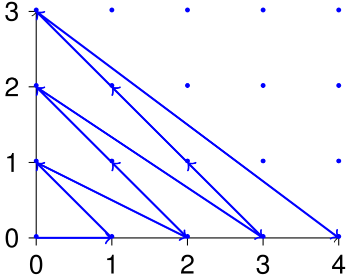

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Same size as the natural numbers!!

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The rationals are countably infinite.

Real numbers..

Real numbers are same size as integers?

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Construct "diagonal" number: .7

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Construct “diagonal” number: .77

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If reals are countable then so is $[0, 1]$.

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5. Show that t is in S.

Diagonalization.

1. Assume that a set S can be enumerated.
2. Consider an arbitrary list of all the elements of S.
3. Use the diagonal from the list to construct a new element t.
4. Show that t is different from all elements in the list
   $\implies$ t is not in the list.
5. Show that t is in S.
6. Contradiction.

Another diagonalization.

The set of all subsets of N.

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Example subsets of N: $\{0\}$,

Another diagonalization.

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There is a listing, L, that contains all subsets of N.

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Theorem: The set of all subsets of N is not countable.

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Theorem: The set of all subsets of N is not countable.
(The set of all subsets of S, is the **powerset** of N.)

Diagonalize Natural Number.

Natural numbers have a listing, *L*.

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Make a diagonal number, D:
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“Diagonal number construction” requires an infinite number of digits.

The Continuum hypothesis.

There is no set with cardinality between the naturals and the reals.

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There is no set with cardinality between the naturals and the reals.

First of Hilbert's problems!

Cardinalities of uncountable sets?

Cardinality of $[0, 1]$ smaller than all the reals?

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$$f(x) = \begin{cases} x + \frac{1}{2} & 0 \leq x \leq 1/2 \\ \frac{1}{4x} & x > 1/2 \end{cases}$$

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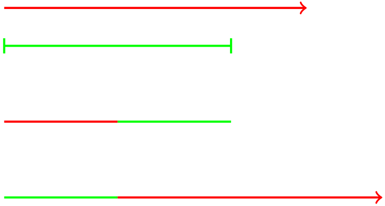

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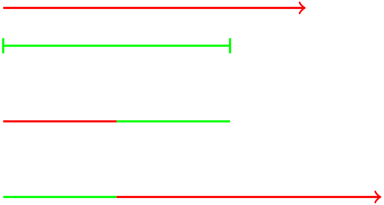

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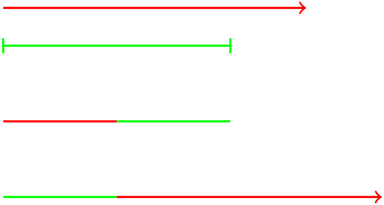

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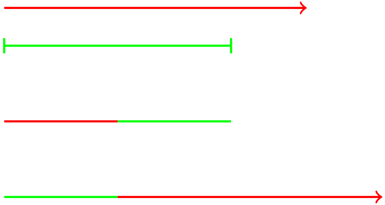

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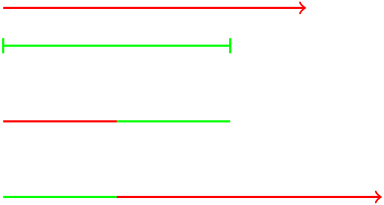

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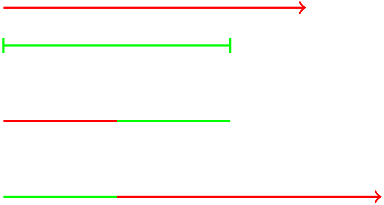

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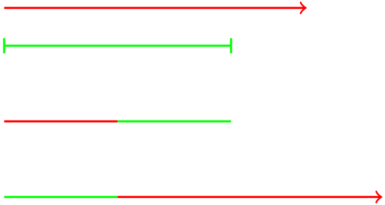

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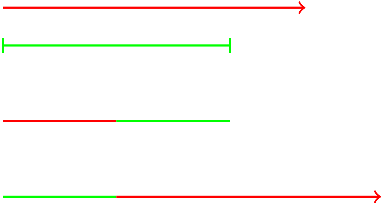

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$[0,1]$ is same cardinality as nonnegative reals!

Generalized Continuum hypothesis.

There is no infinite set whose cardinality is between the cardinality of an infinite set and its power set.

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There is no infinite set whose cardinality is between the cardinality of an infinite set and its power set.

The powerset of a set is the set of all subsets.

Resolution of hypothesis?

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Uh oh....