## Refresh: Counting.

First Rule of counting: Objects from a sequence of choices:
$n_i$ possibilitities for $i$th choice.
$n_1 \times n_2 \times \cdots \times n_k$ objects.

Second Rule of counting: If order does not matter.
Count with order. Divide by number of orderings/sorted object.
Typically: $\binom{n}{k}$.

Stars and Bars: Sample $k$ objects with replacement from $n$.
Order doesn't matter. $k$ stars $n-1$ bars.
Typically: $\binom{n+k-1}{k}$ or $\binom{n+k-1}{n-1}$.

Inclusion/Exclusion: two sets of objects.
Add number of each and then subtract intersection of sets.
Sum Rule: If disjoint just add.

Combinatorial Proofs: Identity from counting same in two ways.
Pascal's Triangle Example: $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.
RHS: Number of subsets of $n+1$ items size $k$.
LHS: $\binom{n}{k-1}$ counts subsets of $n+1$ items with first item.
$\binom{n}{k}$ counts subsets of $n+1$ items without first item.
Disjoint – so add!

## CS70: On to probability.

Modeling Uncertainty: Probability Space

1. Key Points
2. Random Experiments
3. Probability Space

## Key Points

- Uncertainty does not mean "nothing is known"
- How to best make decisions under uncertainty?
  - Buy stocks
  - Detect signals (transmitted bits, speech, images, radar, diseases, etc.)
  - Control systems (Internet, airplane, robots, self-driving cars, schedule surgeries in a hospital, etc.)
- How to best use 'artificial' uncertainty?
  - Play games of chance
  - Design randomized algorithms.
- Probability
  - Models knowledge about uncertainty
  - Optimizes use of knowledge to make decisions

## The Magic of Probability

Uncertainty: vague, fuzzy, confusing, scary, hard to think about.

Probability:
Precise, unambiguous, simple(!) way to reason about uncertainty.

Uncertainty = Fear


Probability = Serenity

Our mission: help you discover the serenity of Probability, i.e., enable you to think clearly about uncertainty.

Your cost: focused attention and practice on examples and problems.

## Random Experiment: Flip one Fair Coin

Flip a fair coin: (*One flips or tosses a coin*)


- Possible outcomes: Heads ($H$) and Tails ($T$)
  (*One flip yields either 'heads' or 'tails'.*)
- Likelihoods: $H$ : 50% and $T$ : 50%

## Random Experiment: Flip one Fair Coin

Flip a fair coin:


What do we mean by the likelihood of tails is 50%?

Two interpretations:

- Single coin flip: 50% chance of 'tails' [subjectivist]

  *Willingness to bet on the outcome of a single flip*
- Many coin flips: About half yield 'tails' [frequentist]

  *Makes sense for many flips*
- Question: Why does the fraction of tails converge to the same value every time? Statistical Regularity! Deep!

## Random Experiment: Flip one Fair Coin

Flip a fair coin: model


Physical Experiment　　Probability Model

- The physical experiment is complex. (Shape, density, initial momentum and position, ...)
- The Probability model is simple:
  - A set $\Omega$ of outcomes: $\Omega = \{H, T\}$.
  - A probability assigned to each outcome: $Pr[H] = 0.5, Pr[T] = 0.5$.

## Random Experiment: Flip one Unfair Coin

Flip an unfair (biased, loaded) coin:


- Possible outcomes: Heads ($H$) and Tails ($T$)
- Likelihoods: $H : p \in (0,1)$ and $T : 1 - p$
- Frequentist Interpretation:

  Flip many times $\Rightarrow$ Fraction $1 - p$ of tails
- Question: How can one figure out $p$? Flip many times
- Tautology? No: Statistical regularity!

## Random Experiment: Flip one Unfair Coin

Flip an unfair (biased, loaded) coin: model


Physical Experiment　　Probability Model

## Flip Two Fair Coins

- Possible outcomes: $\{HH, HT, TH, TT\} \equiv \{H, T\}^2$.
- Note: $A \times B := \{(a, b) \mid a \in A, b \in B\}$ and $A^2 := A \times A$.
- Likelihoods: $1/4$ each.


## Flip Glued Coins

Flips two coins glued together side by side:


Glued coins

- Possible outcomes: $\{HT, TH\}$.
- Likelihoods: $HT : 0.5, TH : 0.5$.
- Note: Coins are glued so that they show different faces.

## Flip two Attached Coins

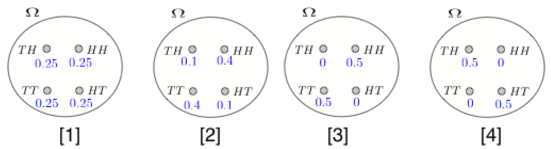
Flips two coins attached by a spring:


Attached coins

- Possible outcomes: $\{HH, HT, TH, TT\}$.
- Likelihoods: $HH : 0.4, HT : 0.1, TH : 0.1, TT : 0.4$.
- Note: Coins are attached so that they tend to show the same face, unless the spring twists enough.

## Flipping Two Coins

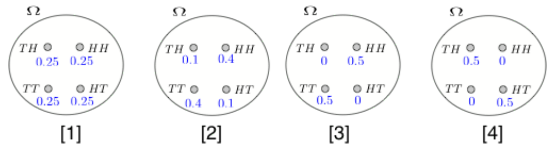
Here is a way to summarize the four random experiments:




- $\Omega$ is the set of *possible* outcomes;
- Each outcome has a probability (likelihood);
- The probabilities are $\geq 0$ and add up to 1;
- Fair coins: [1]; Glued coins: [3], [4];

  Spring-attached coins: [2];

## Flipping Two Coins




Important remarks:

- Each outcome describes the two coins.
- E.g., $HT$ is one outcome of each of the above experiments.
- Wrong to think that outcomes are $\{H, T\}$ and that one picks twice from that set.
- Indeed, this viewpoint misses the relationship between the two flips.
- Each $\omega \in \Omega$ describes one outcome of the complete experiment.
- $\Omega$ and the probabilities specify the random experiment.

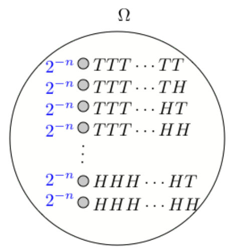
## Flipping $n$ times

Flip a fair coin $n$ times (some $n \geq 1$):

- Possible outcomes: $\{TT\cdots T, TT\cdots H, \ldots, HH\cdots H\}$.

  Thus, $2^n$ possible outcomes.
- Note: $\{TT\cdots T, TT\cdots H, \ldots, HH\cdots H\} = \{H, T\}^n$.

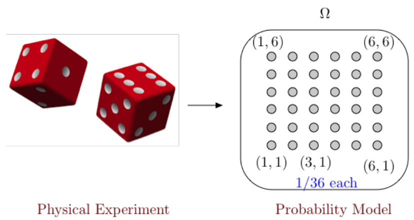
  $A^n := \{(a_1, \ldots, a_n) \mid a_1 \in A, \ldots, a_n \in A\}$. $|A^n| = |A|^n$.
- Likelihoods: $1/2^n$ each.




## Roll two Dice

Roll a balanced 6-sided die twice:

- Possible outcomes: $\{1, 2, 3, 4, 5, 6\}^2 = \{(a, b) \mid 1 \leq a, b \leq 6\}$.
- Likelihoods: $1/36$ for each.



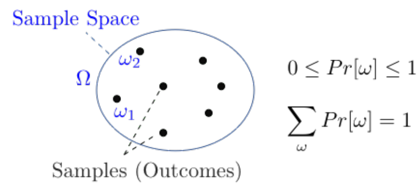

## Probability Space.

1. A "random experiment":
   (a) Flip a biased coin;
   (b) Flip two fair coins;
   (c) Deal a poker hand.
2. A set of possible outcomes: $\Omega$.
   (a) $\Omega = \{H, T\}$;
   (b) $\Omega = \{HH, HT, TH, TT\}$; $|\Omega| = 4$;
   (c) $\Omega = \{ \underline{A\spadesuit\ A\diamondsuit\ A\clubsuit\ A\heartsuit\ K\spadesuit}, \underline{A\spadesuit\ A\diamondsuit\ A\clubsuit\ A\heartsuit\ Q\spadesuit}, \ldots \}$
   $|\Omega| = \binom{52}{5}$.
3. Assign a probability to each outcome: $Pr : \Omega \to [0, 1]$.
   (a) $Pr[H] = p, Pr[T] = 1 - p$ for some $p \in [0, 1]$
   (b) $Pr[HH] = Pr[HT] = Pr[TH] = Pr[TT] = \frac{1}{4}$
   (c) $Pr[\ \underline{A\spadesuit\ A\diamondsuit\ A\clubsuit\ A\heartsuit\ K\spadesuit}\ ] = \cdots = 1/\binom{52}{5}$

## Probability Space: formalism.

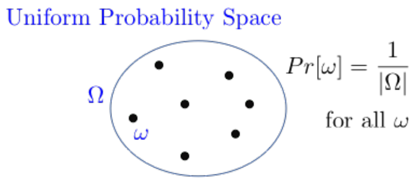
$\Omega$ is the **sample space.**
$\omega \in \Omega$ is a **sample point**. (Also called an **outcome**.)
Sample point $\omega$ has a probability $Pr[\omega]$ where

- $0 \leq Pr[\omega] \leq 1$;
- $\sum_{\omega \in \Omega} Pr[\omega] = 1$.



## Probability Space: Formalism.

In a **uniform probability space** each outcome $\omega$ is equally probable: $Pr[\omega] = \frac{1}{|\Omega|}$ for all $\omega \in \Omega$.

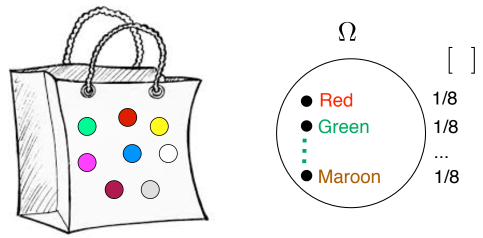



Examples:

- Flipping two fair coins, dealing a poker hand are uniform probability spaces.
- Flipping a biased coin is not a uniform probability space.

## Probability Space: Formalism

Simplest physical model of a uniform probability space:




Physical experiment    Probability model

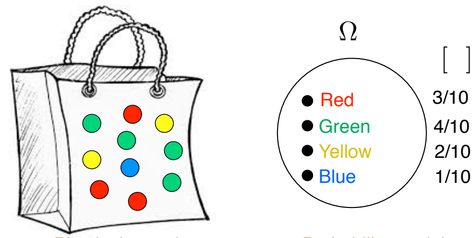
A bag of identical balls, except for their color (or a label). If the bag is well shaken, every ball is equally likely to be picked.

$$\Omega = \{\text{white, red, yellow, grey, purple, blue, maroon, green}\}$$

$$Pr[\text{blue}] = \frac{1}{8}.$$

## Probability Space: Formalism

Simplest physical model of a non-uniform probability space:




Physical experiment    Probability model

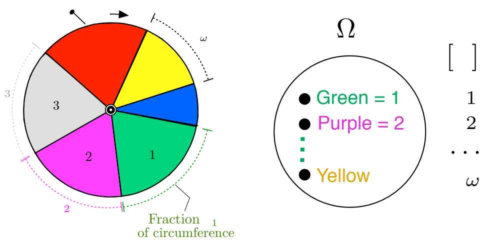
$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$

$$Pr[\text{Red}] = \frac{3}{10}, Pr[\text{Green}] = \frac{4}{10}, \text{ etc.}$$

Note: Probabilities are restricted to rational numbers: $\frac{N_k}{N}$.

## Probability Space: Formalism

Physical model of a general non-uniform probability space:




Physical experiment    Probability model

The roulette wheel stops in sector $\omega$ with probability $p_\omega$.

$$\Omega = \{1, 2, 3, \ldots, N\}, Pr[\omega] = p_\omega.$$

## An important remark

- The random experiment selects one and only one outcome in $\Omega$.
- For instance, when we flip a fair coin twice
  - $\Omega = \{HH, TH, HT, TT\}$
  - The experiment selects *one* of the elements of $\Omega$.
- In this case, its wrong to think that $\Omega = \{H, T\}$ and that the experiment selects two outcomes.
- Why? Because this would not describe how the two coin flips are related to each other.
- For instance, say we glue the coins side-by-side so that they face up the same way. Then one gets $HH$ or $TT$ with probability 50% each. This is not captured by 'picking two outcomes.'

## Summary of Probability Basics

Modeling Uncertainty: Probability Space

1. Random Experiment
2. Probability Space: $\Omega; Pr[\omega] \in [0,1]; \sum_\omega Pr[\omega] = 1$.
3. Uniform Probability Space: $Pr[\omega] = 1/|\Omega|$ for all $\omega \in \Omega$.

## Onwards in Probability.

Events, Conditional Probability, Independence, Bayes' Rule

## CS70: On to Events.

Events, Conditional Probability, Independence, Bayes' Rule

Today: Events.

## Probability Basics Review

**Setup:**

- Random Experiment.
  Flip a fair coin twice.
- Probability Space.
  - **Sample Space:** Set of outcomes, $\Omega$.
    $\Omega = \{HH, HT, TH, TT\}$
    (Note: Not $\Omega = \{H, T\}$ with two picks!)
  - **Probability:** $Pr[\omega]$ for all $\omega \in \Omega$.
    $Pr[HH] = \cdots = Pr[TT] = 1/4$
    1. $0 \le Pr[\omega] \le 1$.
    2. $\sum_{\omega \in \Omega} Pr[\omega] = 1$.

## Set notation review

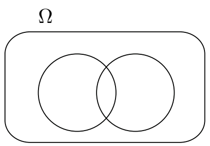



Figure: Two events

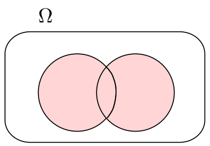



Figure: Union (or)

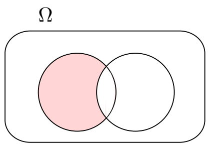



Figure: Difference ($A$, not $B$)

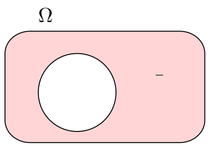



Figure: Complement (not)

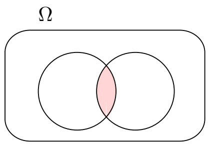



Figure: Intersection (and)

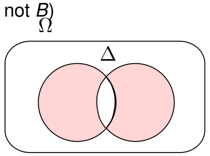



Figure: Symmetric difference (only one)

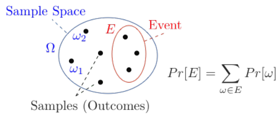
## Probability of exactly one 'heads' in two coin flips?

Idea: Sum the probabilities of all the different outcomes that have exactly one 'heads': $HT, TH$.

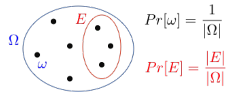
This leads to a definition!

**Definition:**

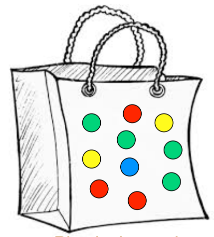
- An **event,** $E$**,** is a subset of outcomes: $E \subset \Omega$.
- The **probability of** $E$ is defined as $Pr[E] = \sum_{\omega \in E} Pr[\omega]$.



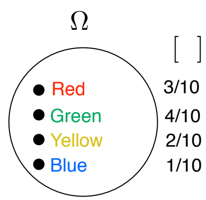

Uniform Probability Space




## Event: Example



Physical experiment




Probability model

$\Omega = \{\text{Red, Green, Yellow, Blue}\}$

$Pr[\text{Red}] = \frac{3}{10}, Pr[\text{Green}] = \frac{4}{10}$, etc.

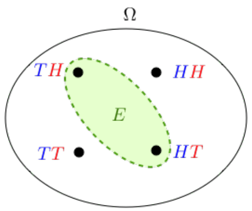
$E = \{Red, Green\} \Rightarrow Pr[E] = \frac{3+4}{10} = \frac{3}{10} + \frac{4}{10} = Pr[\text{Red}] + Pr[\text{Green}]$.

## Probability of exactly one heads in two coin flips?

Sample Space, $\Omega = \{HH, HT, TH, TT\}$.

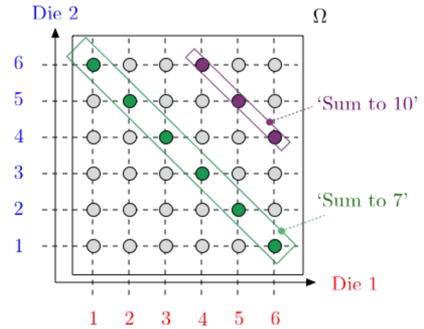
Uniform probability space: $Pr[HH] = Pr[HT] = Pr[TH] = Pr[TT] = \frac{1}{4}$.

Event, $E$, "exactly one heads": $\{TH, HT\}$.




$$Pr[E] = \sum_{\omega \in E} Pr[\omega] = \frac{|E|}{|\Omega|} = \frac{2}{4} = \frac{1}{2}.$$

## Roll a red and a blue die.




$$Pr[\text{Sum to 7}] = \frac{6}{36} \qquad Pr[\text{Sum to 10}] = \frac{3}{36}$$

## Example: 20 coin tosses.

**20 coin tosses**

Sample space: $\Omega =$ set of 20 fair coin tosses.
$\Omega = \{T, H\}^{20} \equiv \{0,1\}^{20}$; $|\Omega| = 2^{20}$.

- What is more likely?
  - $\omega_1 := (1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)$, or
  - $\omega_2 := (1,0,1,1,0,0,0,1,0,1,0,1,1,0,1,1,1,0,0,0)$?

  Answer: Both are equally likely: $Pr[\omega_1] = Pr[\omega_2] = \frac{1}{|\Omega|}$.
- What is more likely?

  ($E_1$) Twenty Hs out of twenty, or
  ($E_2$) Ten Hs out of twenty?

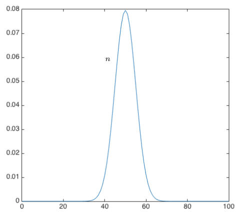
  Answer: Ten Hs out of twenty.

  Why? There are many sequences of 20 tosses with ten Hs; only one with twenty Hs. $\Rightarrow Pr[E_1] = \frac{1}{|\Omega|} \ll Pr[E_2] = \frac{|E_2|}{|\Omega|}$.

$$|E_2| = \binom{20}{10} = 184,756.$$

## Probability of $n$ heads in 100 coin tosses.

$\Omega = \{H, T\}^{100}$; $|\Omega| = 2^{100}$.



Event $E_n$ = '$n$ heads'; $|E_n| = \binom{100}{n}$

$p_n := Pr[E_n] = \frac{|E_n|}{|\Omega|} = \frac{\binom{100}{n}}{2^{100}}$

Observe:

- Concentration around mean: Law of Large Numbers;
- Bell-shape: Central Limit Theorem.

## Exactly 50 heads in 100 coin tosses.

Sample space: $\Omega =$ set of 100 coin tosses $= \{H, T\}^{100}$.
$|\Omega| = 2 \times 2 \times \cdots \times 2 = 2^{100}$.

Uniform probability space: $Pr[\omega] = \frac{1}{2^{100}}$.

Event $E =$ "100 coin tosses with exactly 50 heads"

$|E|$?
Choose 50 positions out of 100 to be heads.
$|E| = \binom{100}{50}$.

$$Pr[E] = \frac{\binom{100}{50}}{2^{100}}.$$

## Calculation.

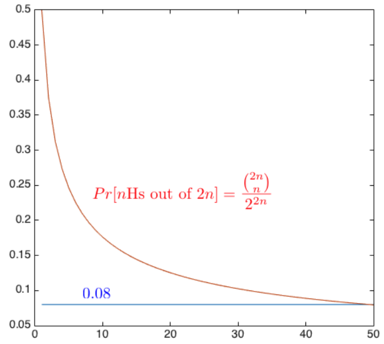
Stirling formula (for large $n$):

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

$$\binom{2n}{n} \approx \frac{\sqrt{4\pi n}(2n/e)^{2n}}{[\sqrt{2\pi n}(n/e)^n]^2} \approx \frac{4^n}{\sqrt{\pi n}}.$$

$$Pr[E] = \frac{|E|}{|\Omega|} = \frac{|E|}{2^{2n}} = \frac{1}{\sqrt{\pi n}} = \frac{1}{\sqrt{50\pi}} \approx .08.$$

# Exactly 50 heads in 100 coin tosses.




# Summary.

1. Random Experiment
2. Probability Space: $\Omega; Pr[\omega] \in [0,1]; \sum_{\omega} Pr[\omega] = 1$.
3. Uniform Probability Space: $Pr[\omega] = 1/|\Omega|$ for all $\omega \in \Omega$.
4. Event: "subset of outcomes." $A \subseteq \Omega$. $Pr[A] = \sum_{w \in A} Pr[\omega]$
5. Some calculations.