Lecture 17: Continuing Probability.

$\boxed{\text{Events, Conditional Probability, Independence, Bayes' Rule}}$

Probability Basics Review

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- Random Experiment.

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    $Pr[HH] = \cdots = Pr[TT] = 1/4$
    1. $0 \leq Pr[\omega] \leq 1.$
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  - **Events.**
    Event $A \subseteq \Omega$, $Pr[A] = \sum_{\omega \in \Omega} Pr[\omega]$.

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(b) Either induction, or argue over sample points.

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Proofs for (a) and (c)? Next...

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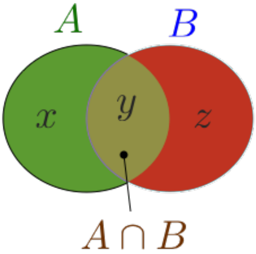

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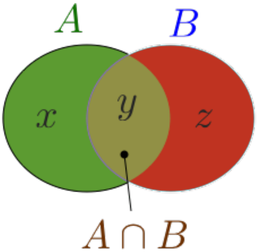

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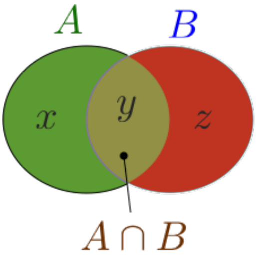

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Another view. Any $\omega \in A \cup B$ is in $A \cap \overline{B}$, $A \cup B$, or $\overline{A} \cap B$. So, add it up.

Roll a Red and a Blue Die.

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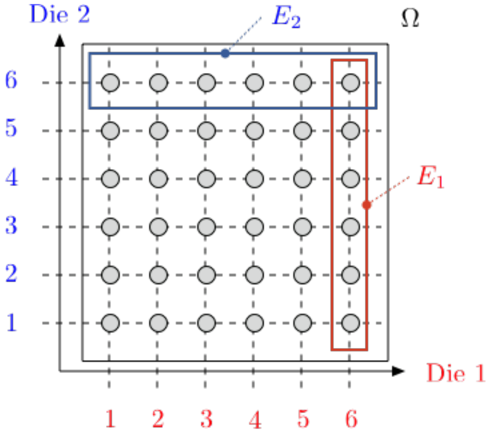

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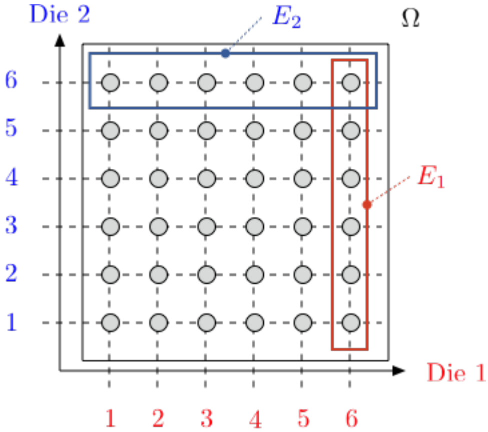

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E_1 = 'Red die shows 6';

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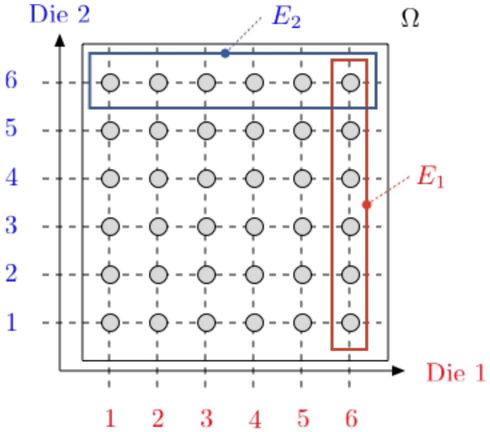

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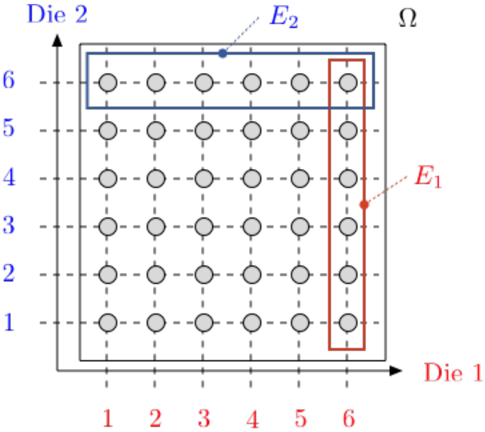

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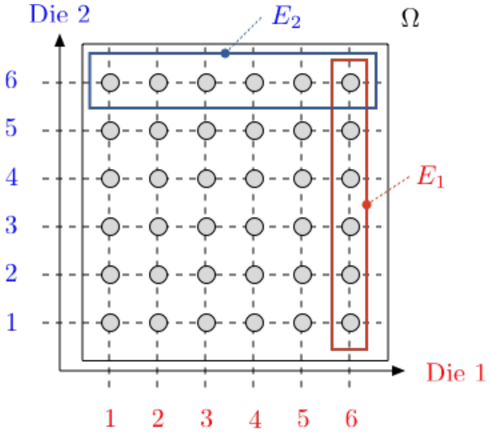

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$$Pr[E_1] = \frac{6}{36},$$

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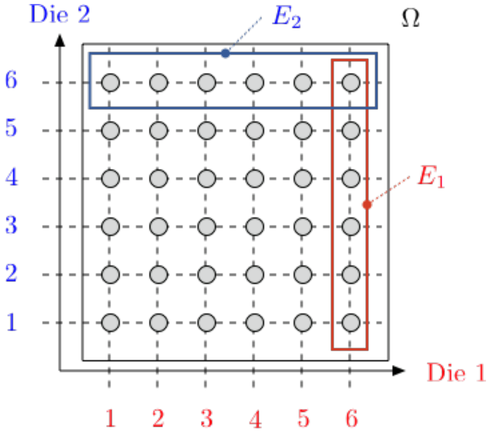

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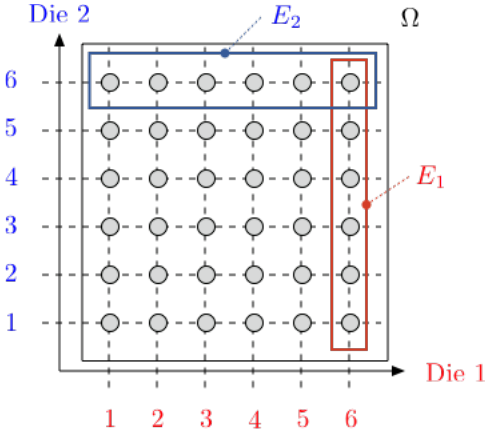

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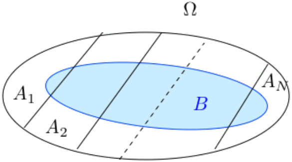
$$Pr[E_1] = \frac{6}{36}, Pr[E_2] = \frac{6}{36}, Pr[E_1 \cup E_2] = \frac{11}{36}.$$

Total probability

Assume that Ω is the union of the disjoint sets $A_1, \ldots, A_N$.



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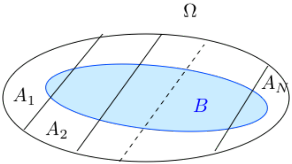

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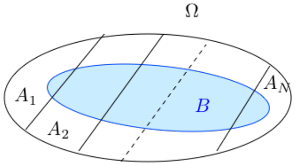

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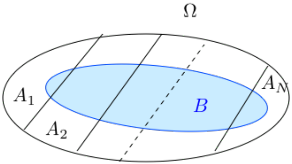

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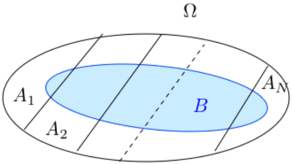

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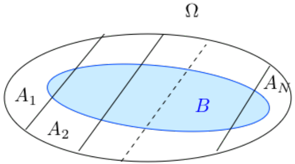

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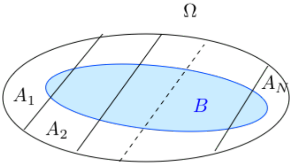

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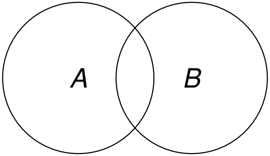

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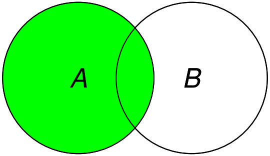

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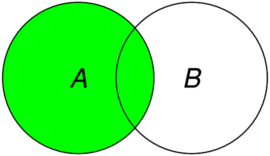

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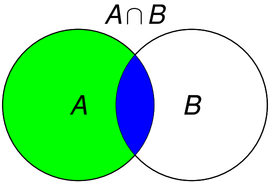
$$Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]}$$



In A!

In B?

Must be in $A \cap B$.

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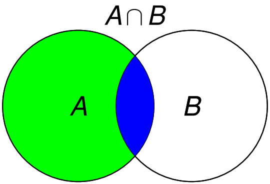

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Conditional probability: example.

Two coin flips.

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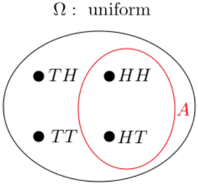

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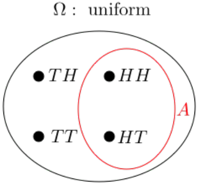

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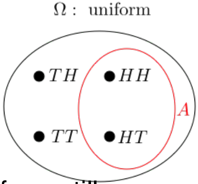

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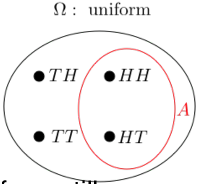

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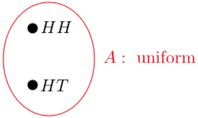

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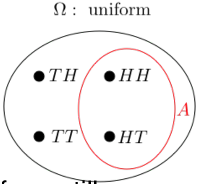

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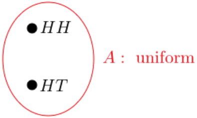

Event $B =$ two heads.

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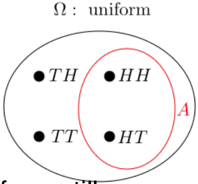

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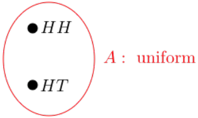

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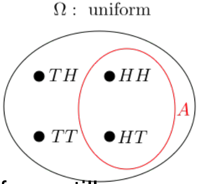

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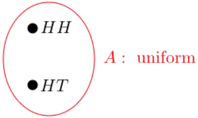

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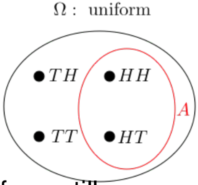

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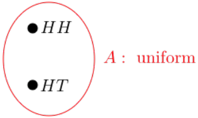

Event $B =$ two heads.

The probability of two heads if the first flip is heads.
The probability of B given A is $1/2$.

A similar example.

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Two coin flips. At least one of the flips is heads.

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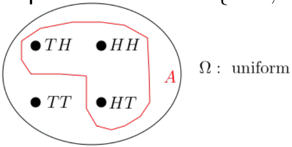

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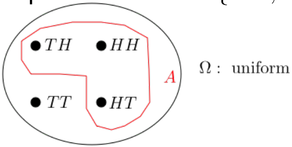

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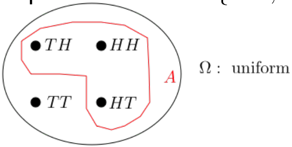

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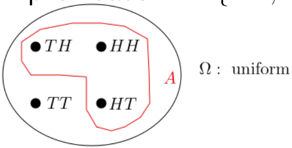

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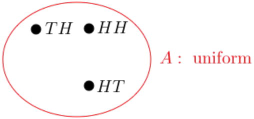

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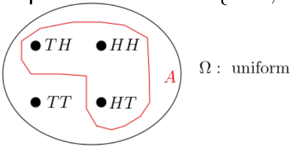

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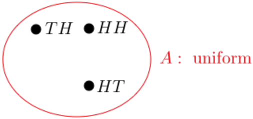

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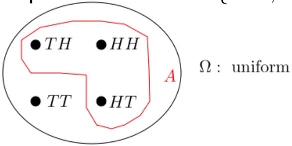

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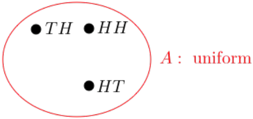

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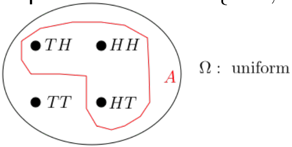

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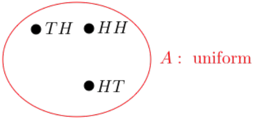

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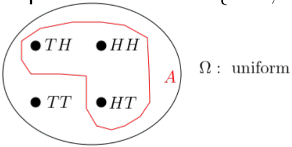

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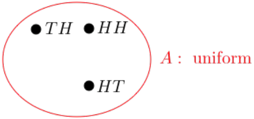

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Conditional Probability: A non-uniform example

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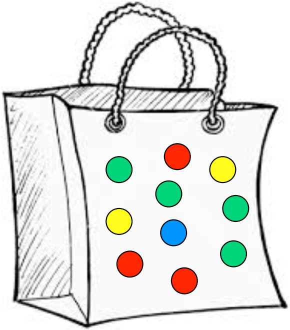
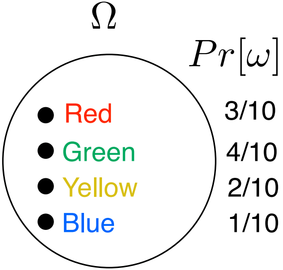


Physical experiment



Probability model

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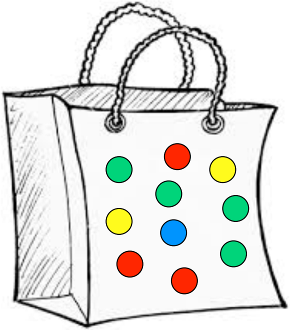


Physical experiment

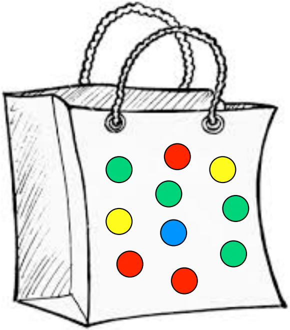
Ω

ω	$Pr[\omega]$
Red	3/10
Green	4/10
Yellow	2/10
Blue	1/10

Probability model

$\Omega = \{$Red, Green, Yellow, Blue$\}$

Conditional Probability: A non-uniform example



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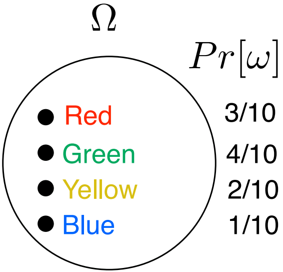

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$\Omega = \{$Red, Green, Yellow, Blue$\}$

Pr[Red|Red or Green] $=$

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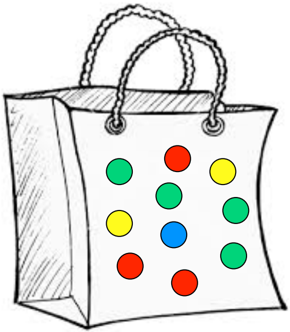

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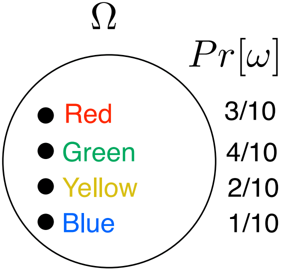
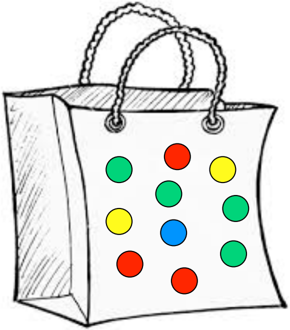


Probability model

$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$

$$Pr[\text{Red}|\text{Red or Green}] = \frac{3}{7} =$$

Conditional Probability: A non-uniform example



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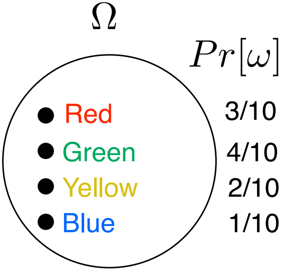

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$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$

$$Pr[\text{Red}|\text{Red or Green}] = \frac{3}{7} = \frac{Pr[\text{Red} \cap (\text{Red or Green})]}{Pr[\text{Red or Green}]}$$

Another non-uniform example

Consider $\Omega = \{1, 2, \ldots, N\}$ with $Pr[n] = p_n$.

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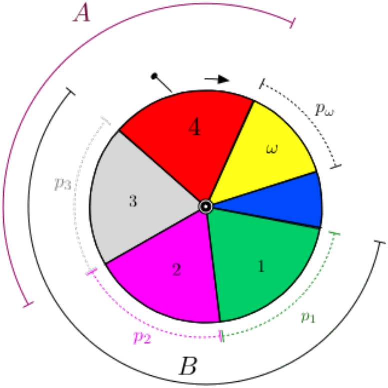

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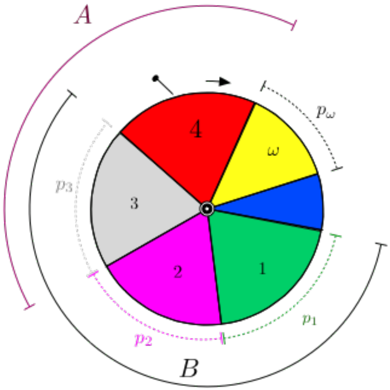

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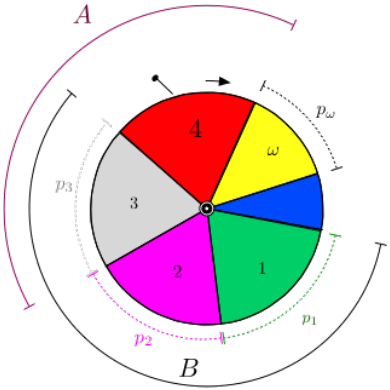

$$Pr[A|B] = \frac{p_3}{p_1 + p_2 + p_3} = \frac{Pr[A \cap B]}{Pr[B]}.$$

Yet another non-uniform example

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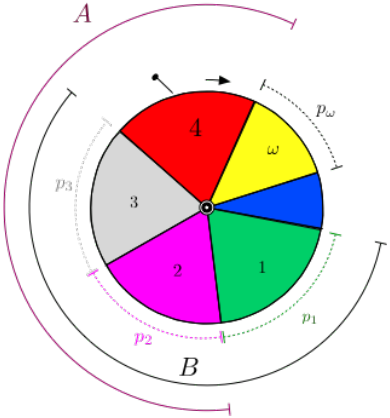

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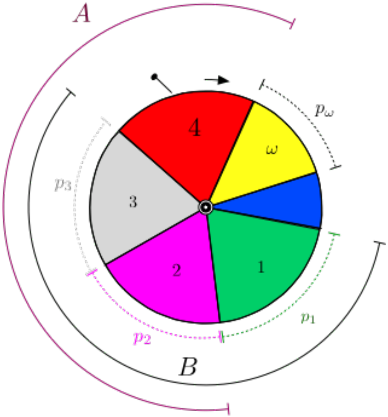

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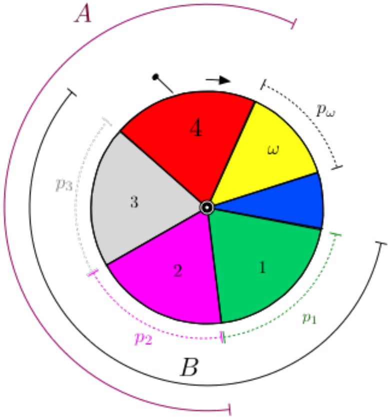

$$Pr[A|B] = \frac{p_2 + p_3}{p_1 + p_2 + p_3} = \frac{Pr[A \cap B]}{Pr[B]}.$$

More fun with conditional probability.

Toss a red and a blue die, sum is 4,

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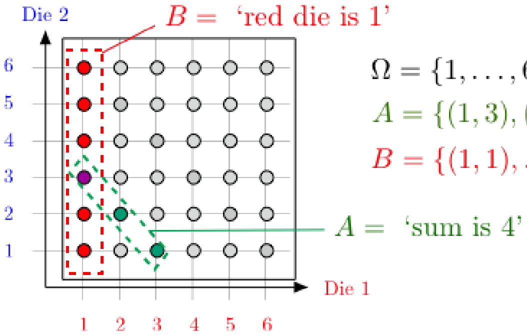

$\Omega = \{1, \ldots, 6\}^2$

$A = \{(1,3),(2,2),(3,1)\}$

$B = \{(1,1), \ldots, (1,6)\}$

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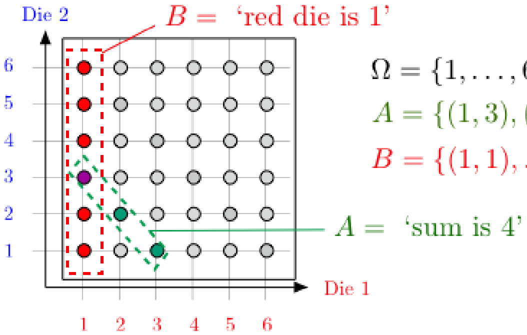

$\Omega = \{1,\ldots,6\}^2$

$A = \{(1,3),(2,2),(3,1)\}$

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$Pr[B|A] = \frac{|B \cap A|}{|A|} = \frac{1}{3};$

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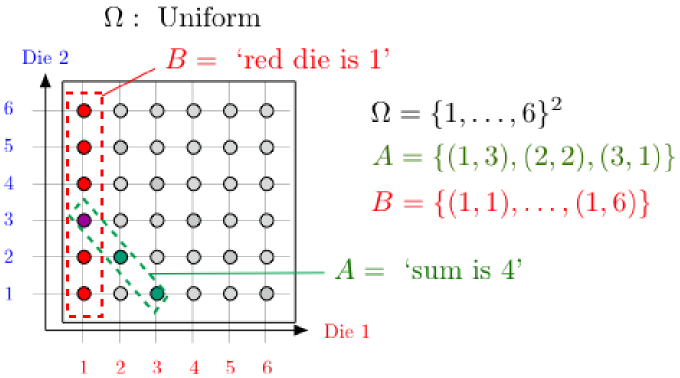

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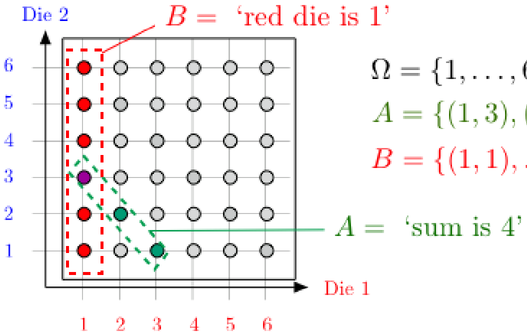

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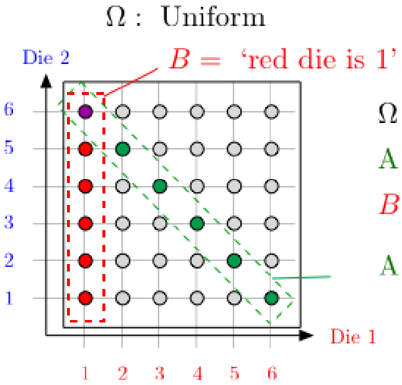

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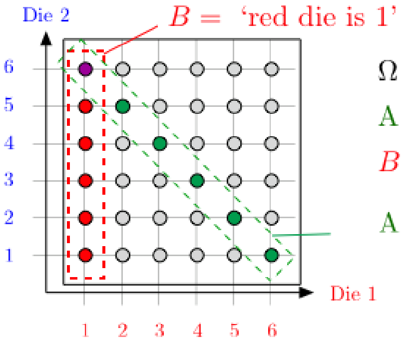

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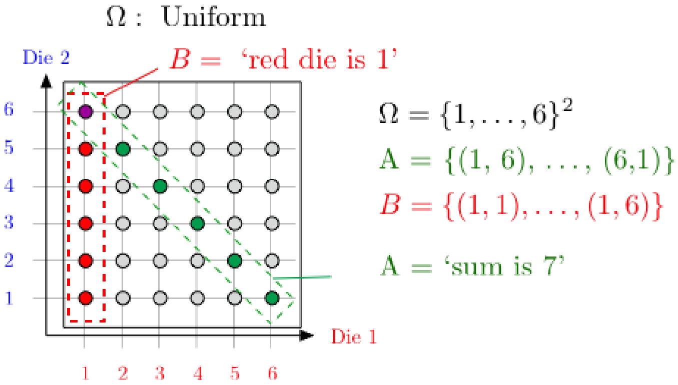

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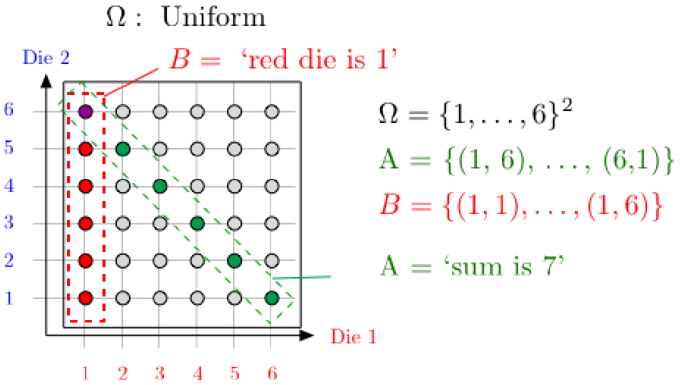

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Observing A does not change your mind about the likelihood of B.

Emptiness..

Suppose I toss 3 balls into 3 bins.

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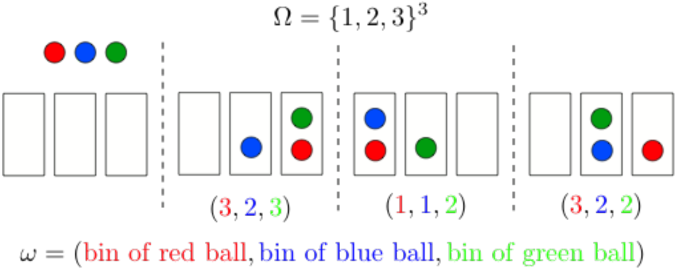

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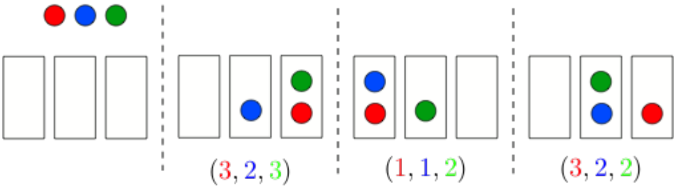

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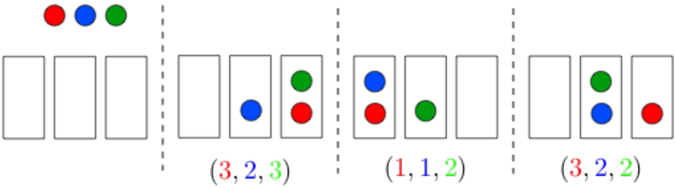

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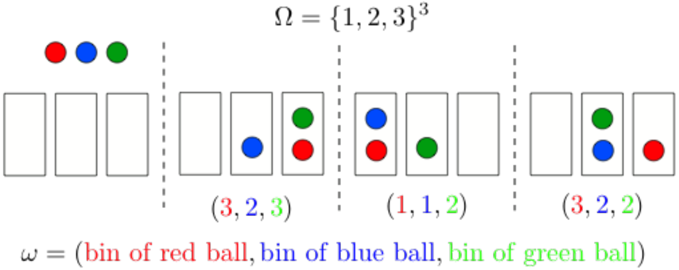

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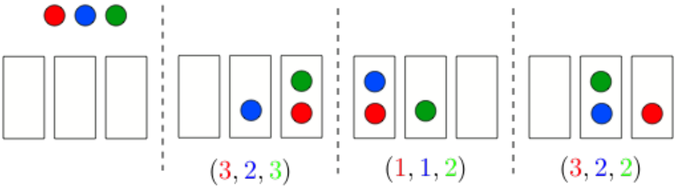

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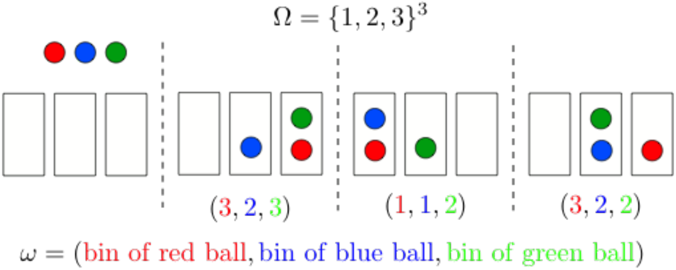

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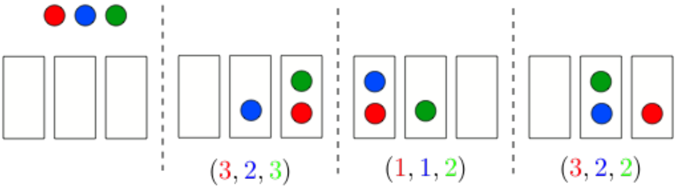

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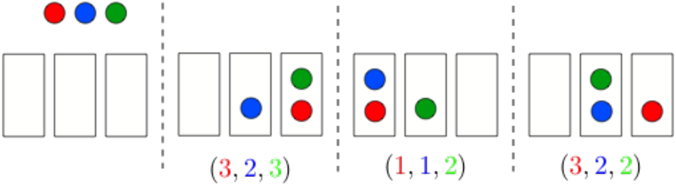

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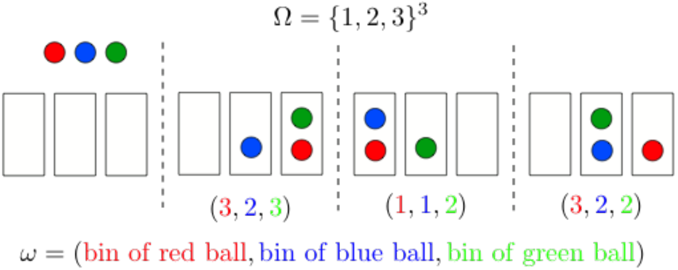

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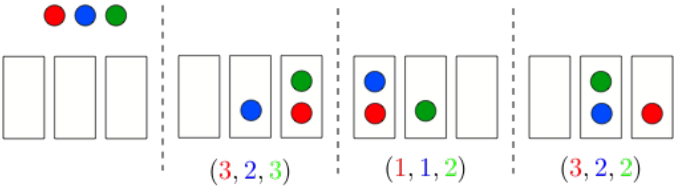

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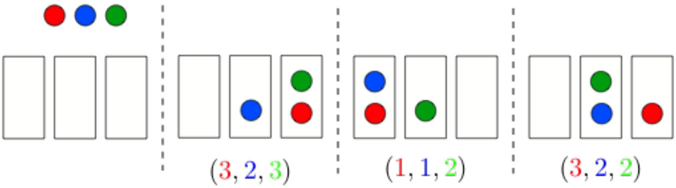

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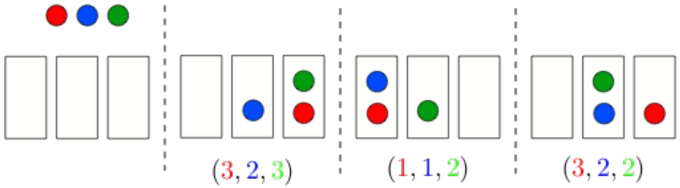

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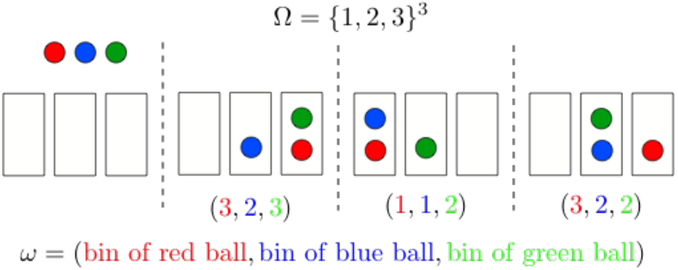

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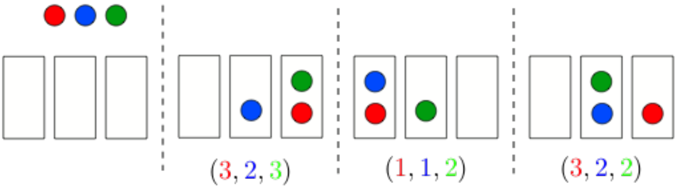

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A is less likely given B: If second bin is empty the first is more likely to have balls in it.

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Flip a fair coin 51 times.

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The likelihood of 51st heads does not depend on the previous flips.

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so that the result holds for $n+1$. $\square$

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Random experiment: Pick a person at random.

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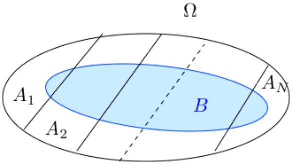
More about such questions later. For fun, check "N. Taleb: Fooled by randomness."

Total probability with Conditional Probability.

Assume that Ω is the union of the disjoint sets $A_1, \ldots, A_N$.



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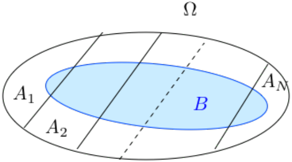

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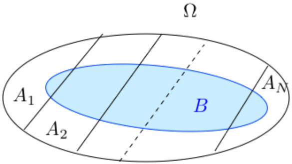

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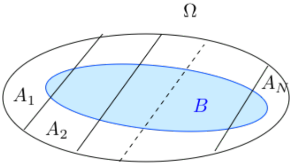

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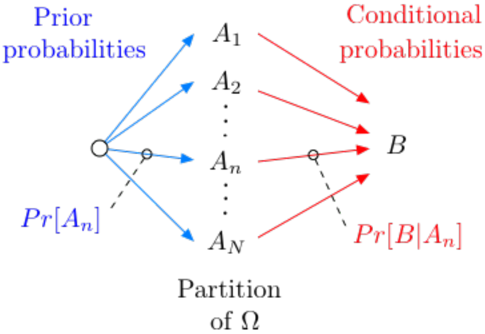

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Lecture basically ended here.