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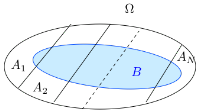
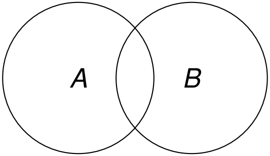


Add it up!

Conditional Probability.

Definition: The **conditional probability** of B given A is

$$Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]}$$



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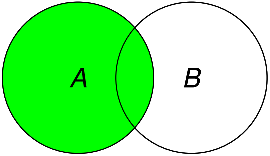

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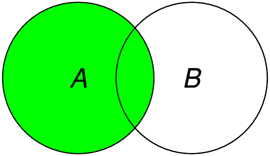

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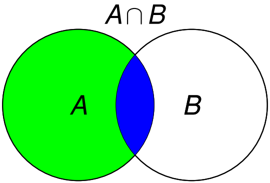

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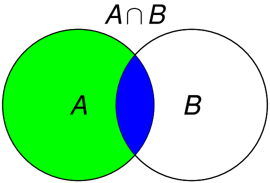

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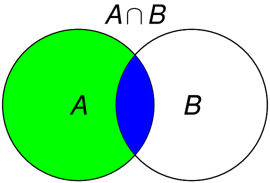

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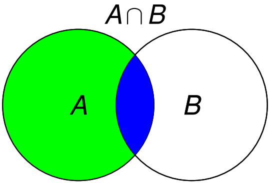

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$$Pr[A_1 \cap \cdots \cap A_n] = Pr[A_1]Pr[A_2|A_1] \cdots Pr[A_n|A_1 \cap \cdots \cap A_{n-1}].$$

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Thus,

$$Pr[A|B] = \frac{Pr[A]Pr[B|A]}{Pr[B]} = \frac{(1/2)(1/2)}{(1/2)(1/2) + (1/2)0.6} \approx 0.45.$$

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- When flipping coins, $A =$ coin 1 yields heads and $B =$ coin 2 yields tails are independent; $Pr[A \cap B] = \frac{1}{4}$, $Pr[A]Pr[B] = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$.

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Definition: Two events A and B are **independent** if

$$Pr[A \cap B] = Pr[A]Pr[B].$$

Examples:

- When rolling two dice, $A =$ sum is 7 and $B =$ red die is 1 are independent; $Pr[A \cap B] = \frac{1}{36}$, $Pr[A]Pr[B] = \left(\frac{1}{6}\right)\left(\frac{1}{6}\right)$.
- When rolling two dice, $A =$ sum is 3 and $B =$ red die is 1 are not independent; $Pr[A \cap B] = \frac{1}{36}$, $Pr[A]Pr[B] = \left(\frac{2}{36}\right)\left(\frac{1}{6}\right)$.
- When flipping coins, $A =$ coin 1 yields heads and $B =$ coin 2 yields tails are independent; $Pr[A \cap B] = \frac{1}{4}$, $Pr[A]Pr[B] = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$.
- When throwing 3 balls into 3 bins, $A =$ bin 1 is empty and $B =$ bin 2 is empty are not independent; $Pr[A \cap B] = \frac{1}{27}$, $Pr[A]Pr[B] = \left(\frac{8}{27}\right)\left(\frac{8}{27}\right)$.

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Conditional Probability: Pictures

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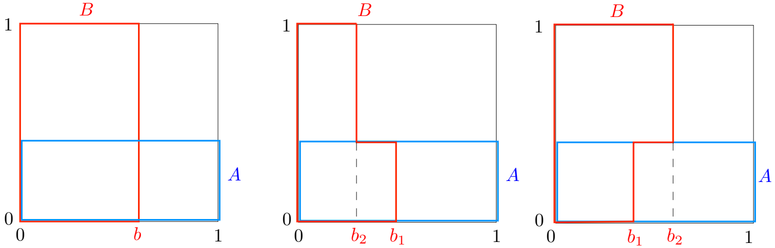
Illustrations: Pick a point uniformly in the unit square



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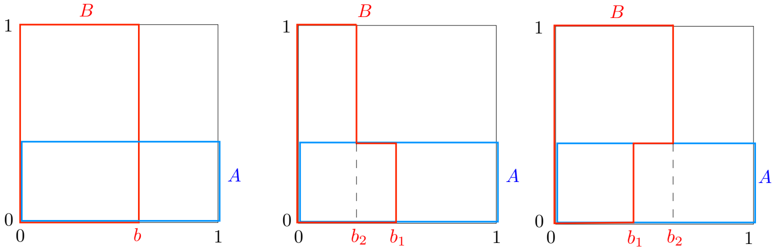

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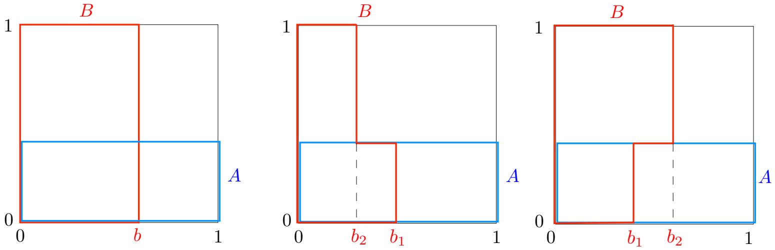

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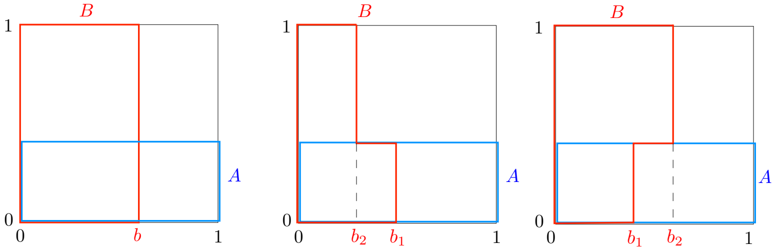

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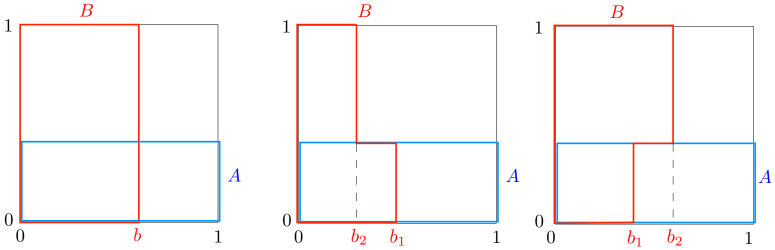

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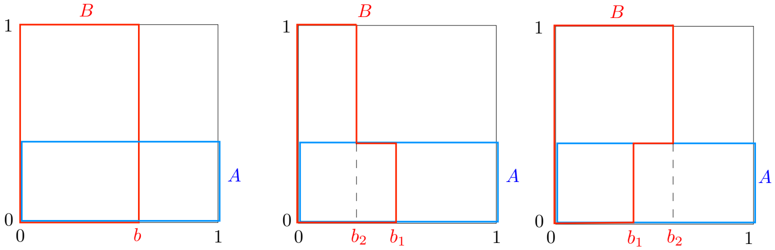

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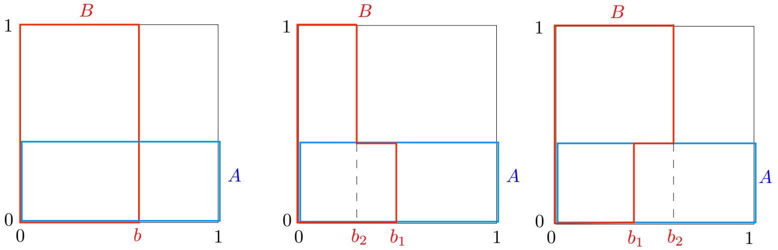

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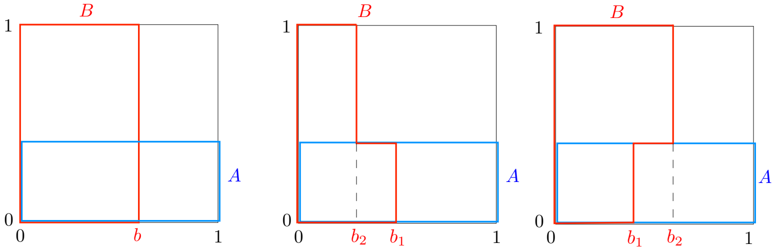

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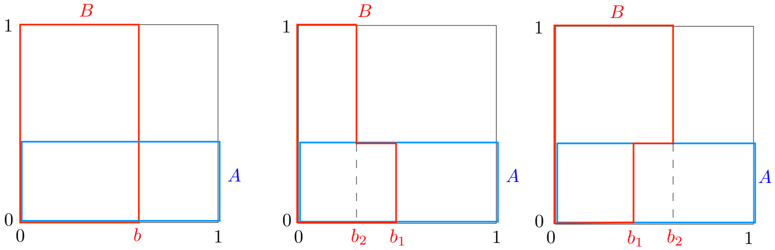

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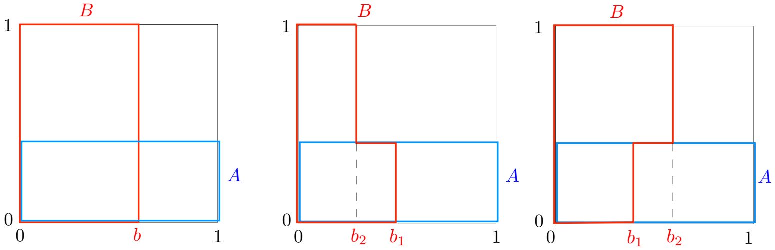

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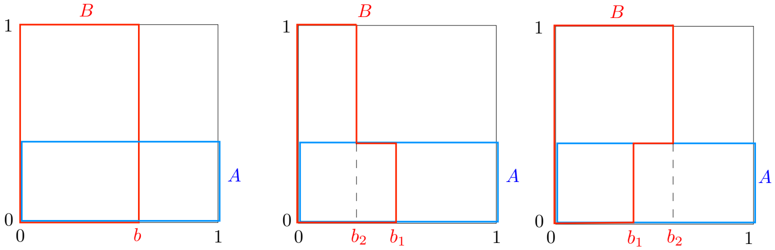

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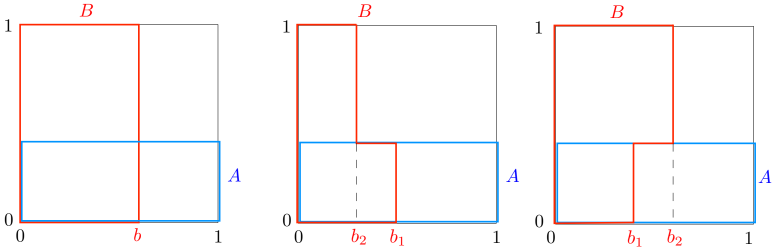

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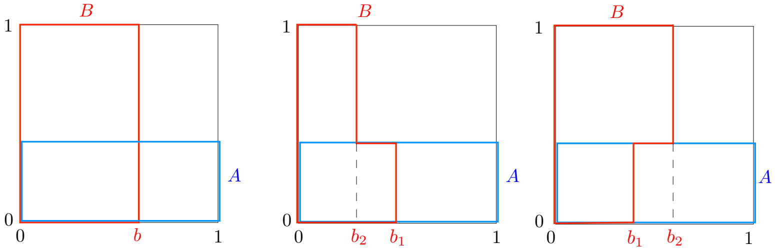

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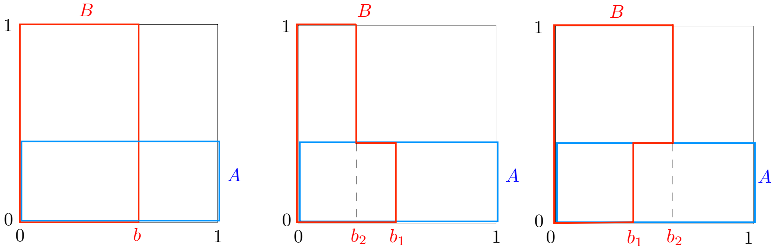

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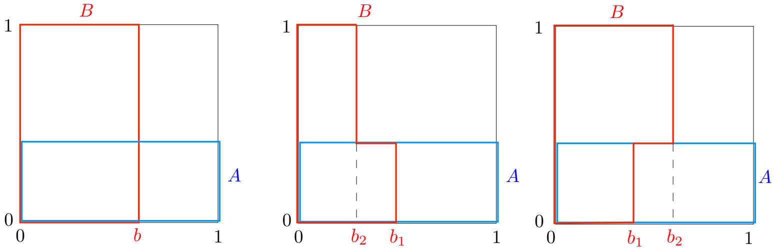

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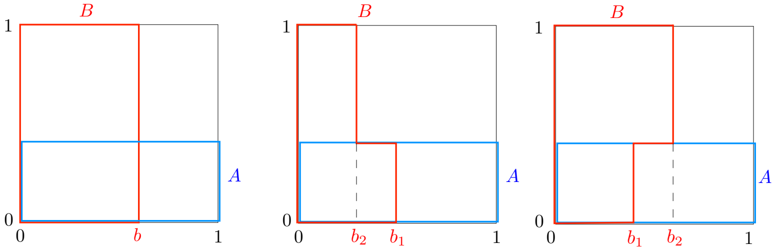

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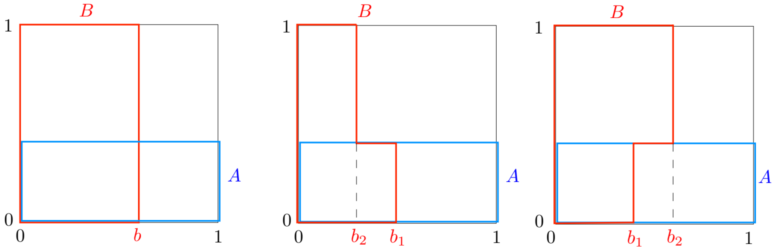

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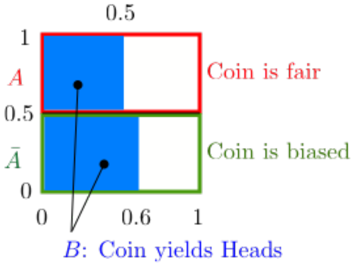

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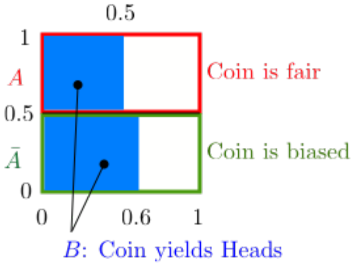

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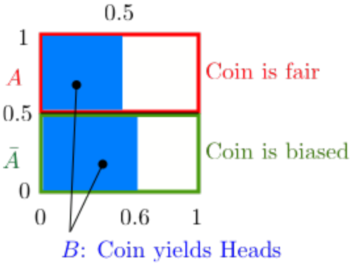

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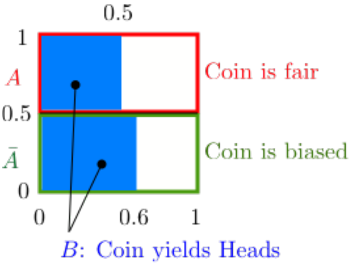

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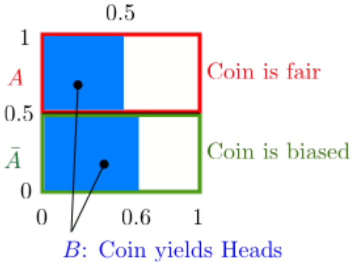

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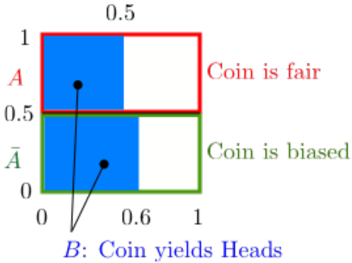

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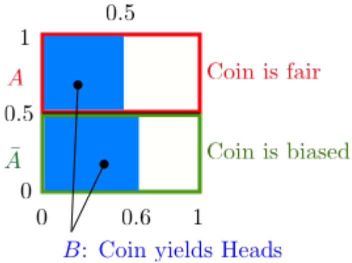

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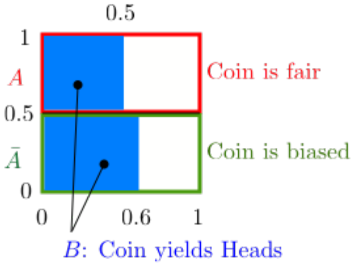

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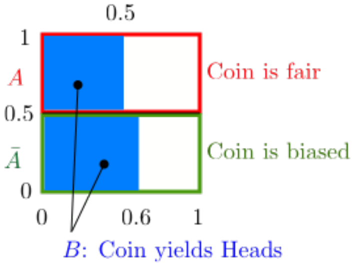

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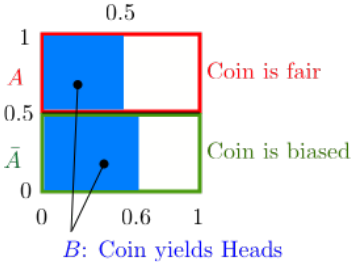

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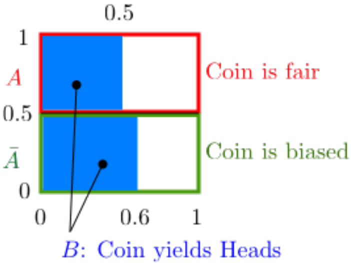

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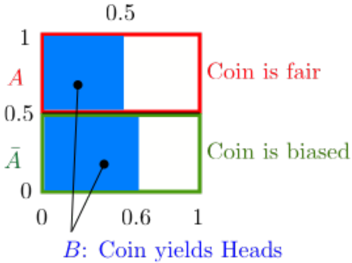
Bayes and Biased Coin



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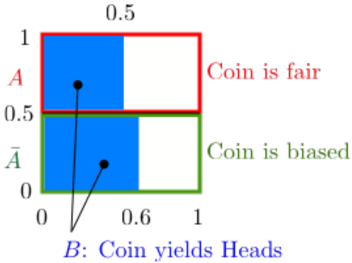

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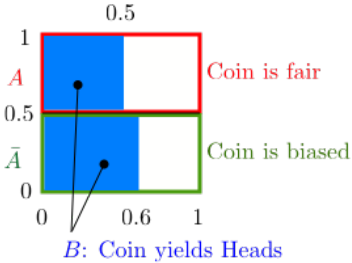

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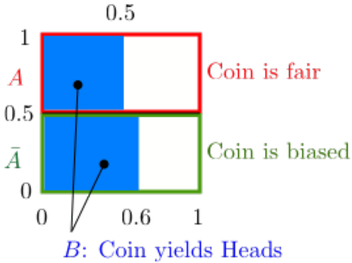

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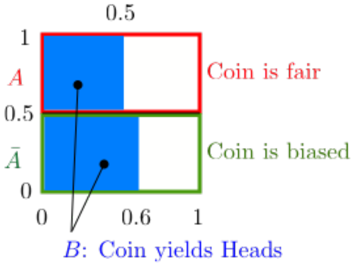

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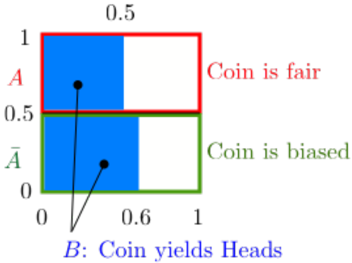
Bayes and Biased Coin



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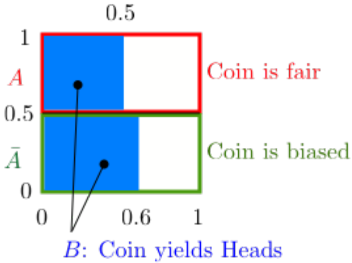

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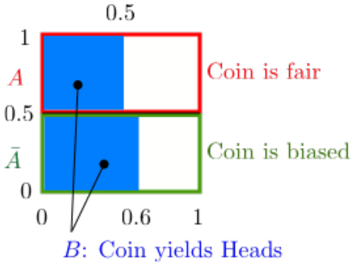
Bayes and Biased Coin



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$$\approx 0.46 = \text{fraction of } B \text{ that is inside } A$$

Bayes: General Case

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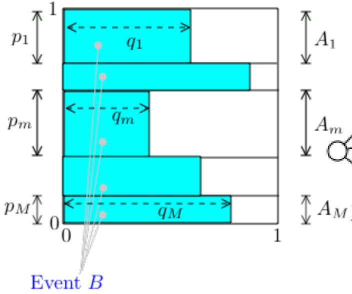

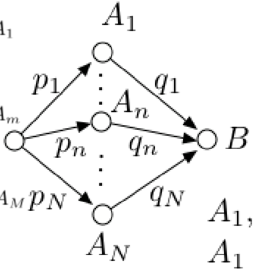

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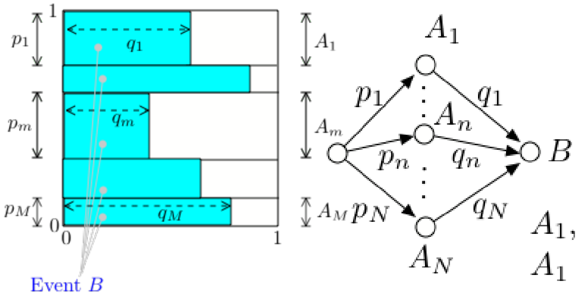

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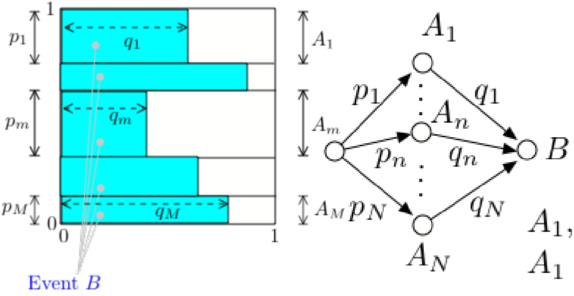

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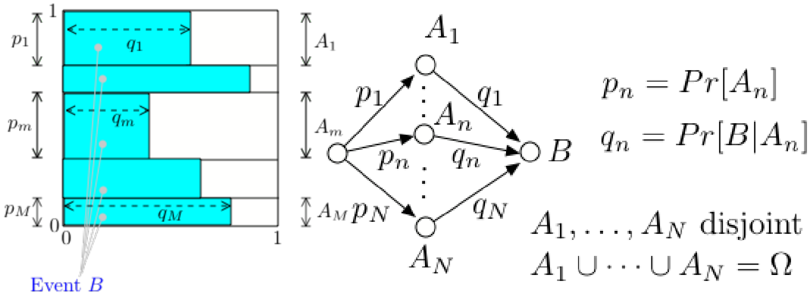

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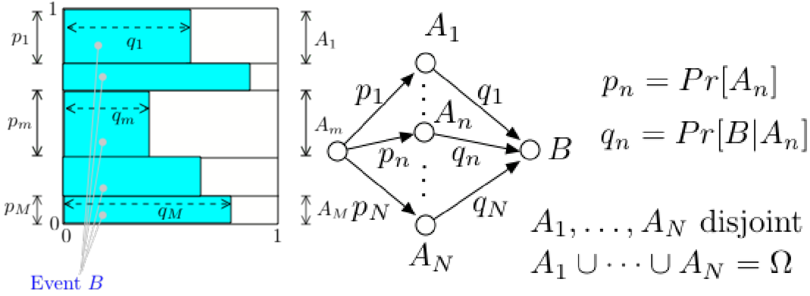

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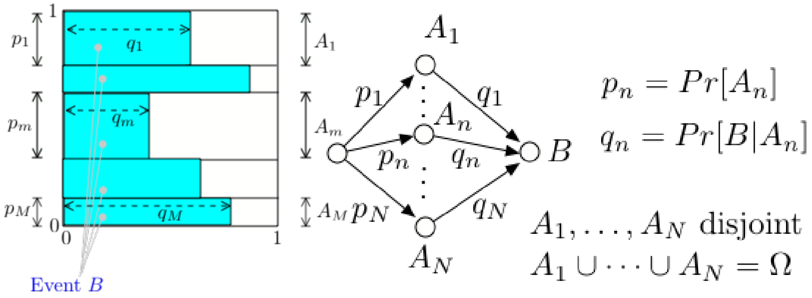

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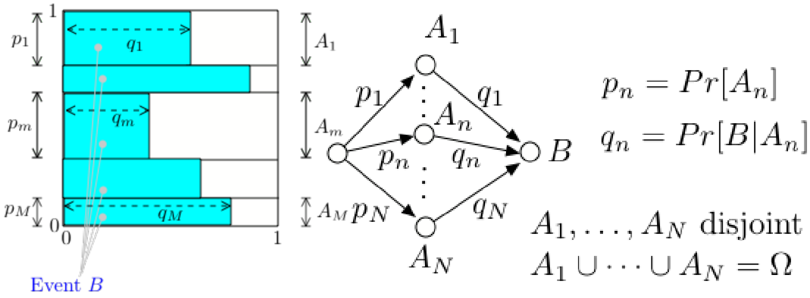

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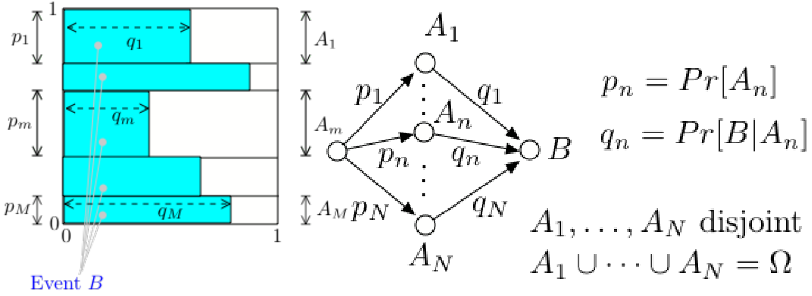

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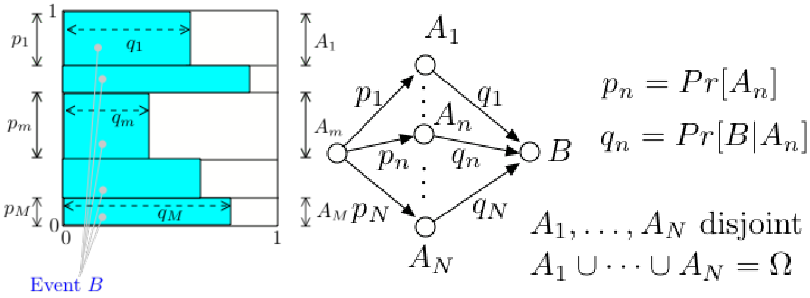

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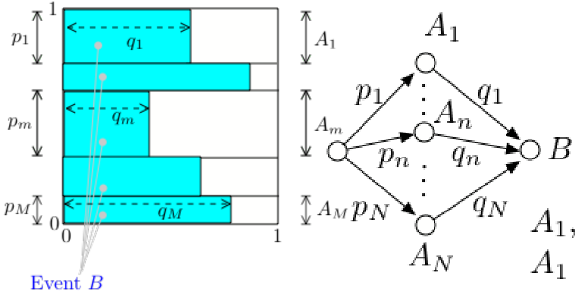

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Bayes Rule

A general picture: We imagine that there are N possible causes $A_1, \ldots, A_N$.

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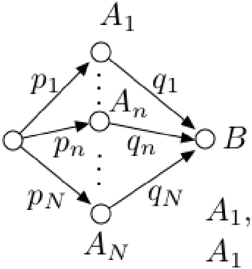

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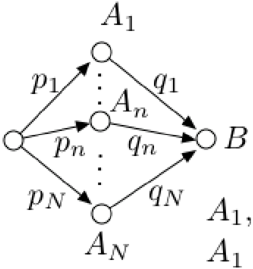

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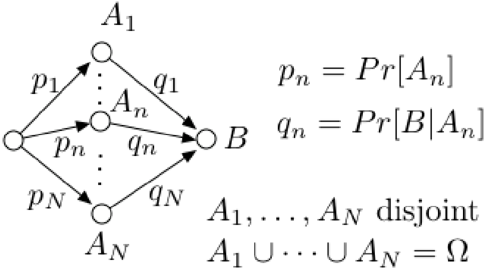

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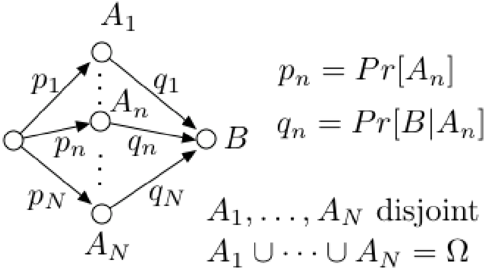

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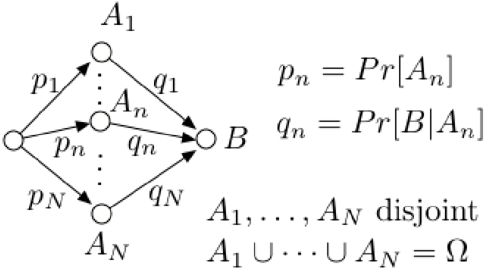

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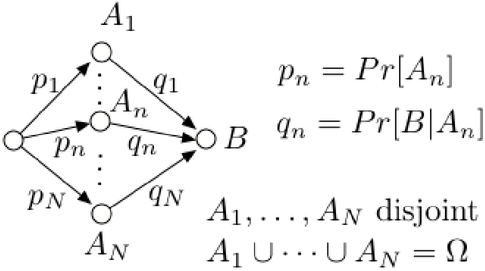

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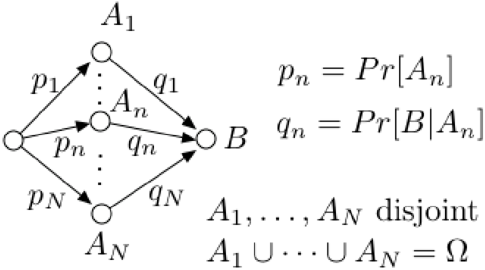

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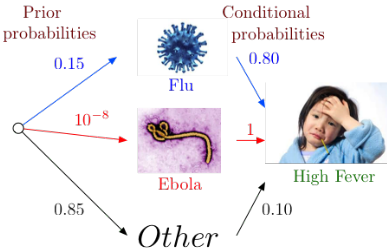
Why do you have a fever?



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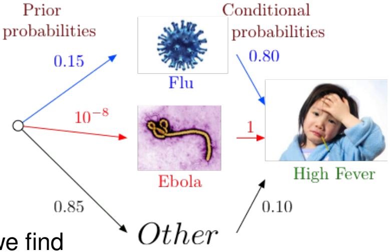

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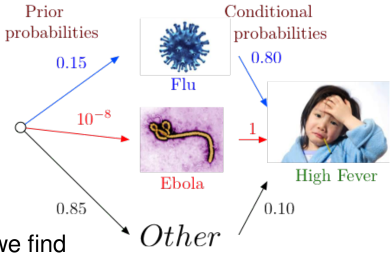

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$$Pr[\text{Flu}|\text{High Fever}] = \frac{0.15 \times 0.80}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 0.58$$

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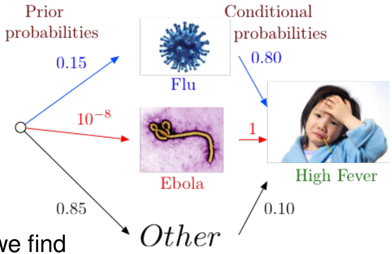

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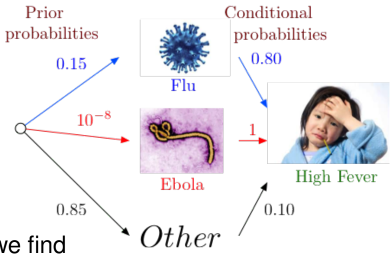

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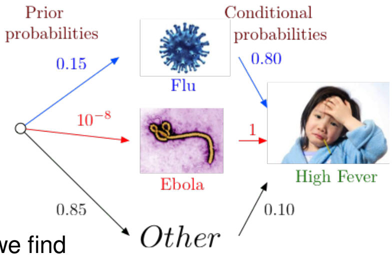
$$Pr[\text{Other}|\text{High Fever}] = \frac{0.85 \times 0.1}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 0.42$$

Why do you have a fever?



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The values $0.58, 5 \times 10^{-8}, 0.42$ are the posterior probabilities.

Why do you have a fever?

Why do you have a fever?

Our “Bayes’ Square” picture:

Why do you have a fever?

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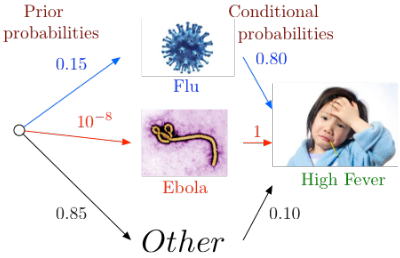

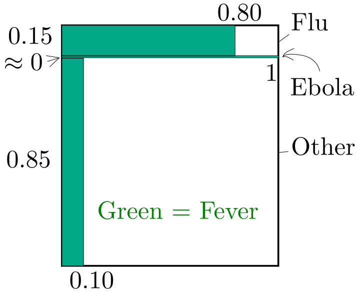

58% of Fever = Flu

≈ 0% of Fever = Ebola

42% of Fever = Other

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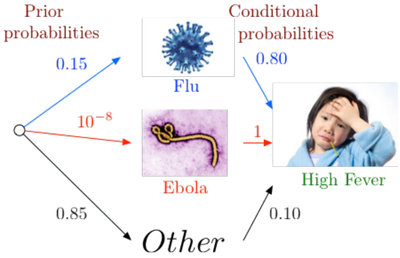

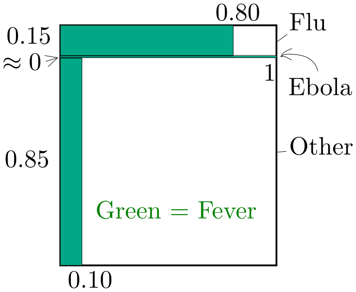

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Note that even though $Pr[\text{Fever}|\text{Ebola}] = 1$,

Why do you have a fever?

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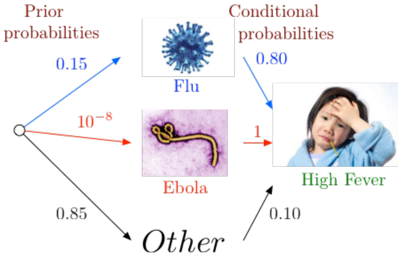

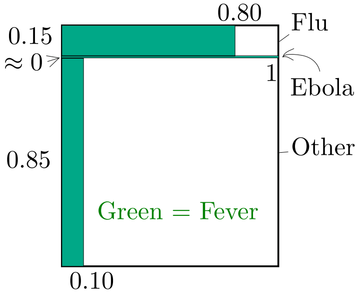

58% of Fever $=$ Flu

$\approx 0\%$ of Fever $=$ Ebola

42% of Fever $=$ Other

Note that even though $Pr[\text{Fever}|\text{Ebola}] = 1$, one has

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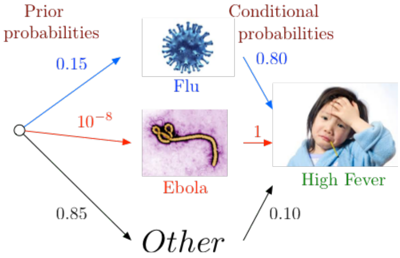

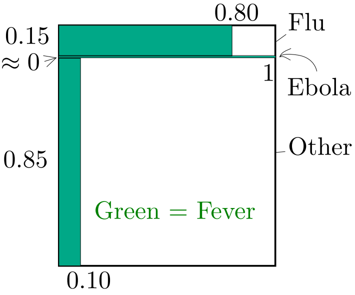

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This example shows the importance of the prior probabilities.

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Bayes' Rule Operations

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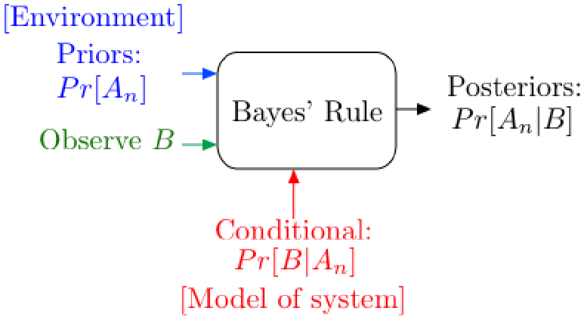

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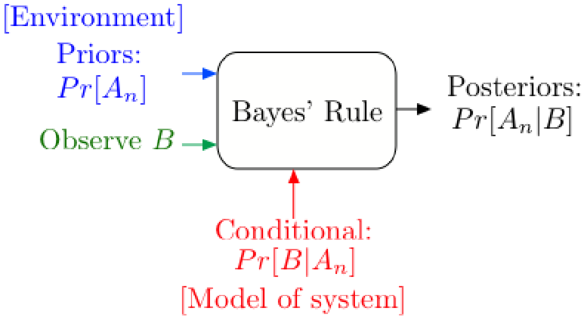

Bayes' Rule: canonical example of how information changes our opinions.

Thomas Bayes

Thomas Bayes

Portrait used of Bayes in a 1936 book,[1] but it is doubtful whether the portrait is actually of him.[2] No earlier portrait or claimed portrait survives.

Born	c. 1701 London, England
Died	7 April 1761 (aged 59) Tunbridge Wells, Kent, England
Residence	Tunbridge Wells, Kent, England
Nationality	English
Known for	Bayes' theorem

Source: Wikipedia.

Thomas Bayes

A

FIG. 3. *Joshua Bayes (1671–1746).*

$\xrightarrow{Pr[B|A]}$

Max over B

B

Thomas Bayes?

A Bayesian picture of Thomas Bayes.

Testing for disease.

Random Experiment: Pick a random male.

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$$Pr[A|B]???$$

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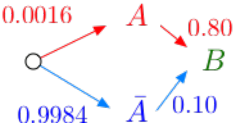

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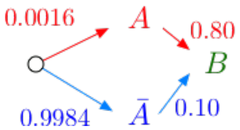

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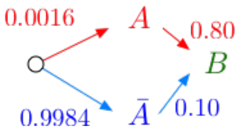

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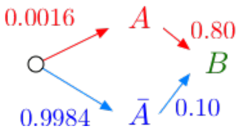

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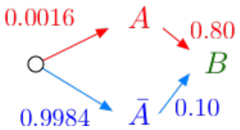

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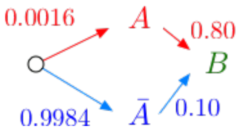

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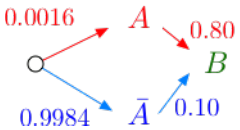

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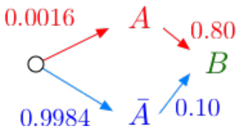

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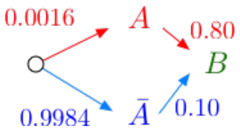

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$\boxed{\text{Events, Conditional Probability, Independence, Bayes' Rule}}$

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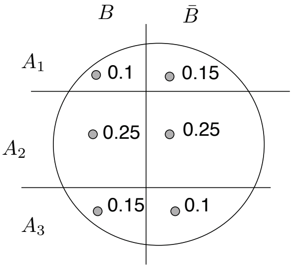

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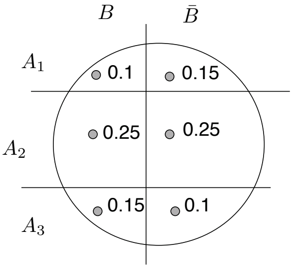

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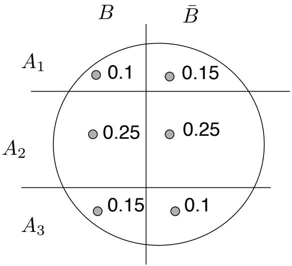

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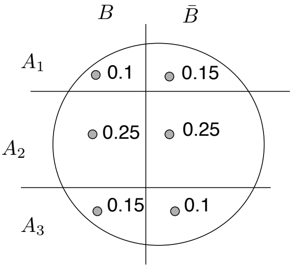

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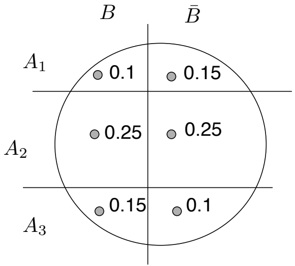

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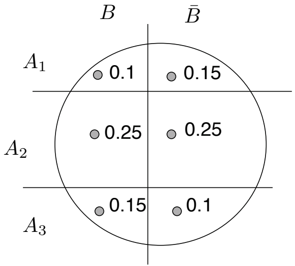

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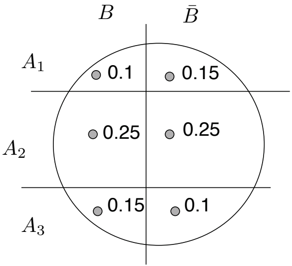

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Flip two fair coins. Let

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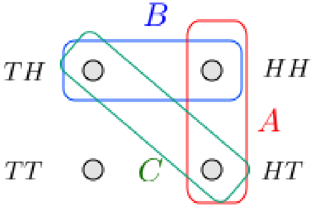

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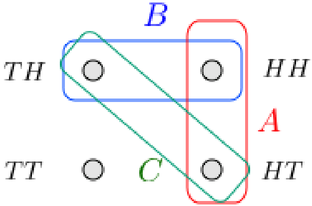

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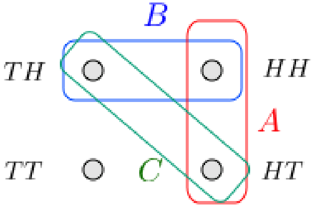

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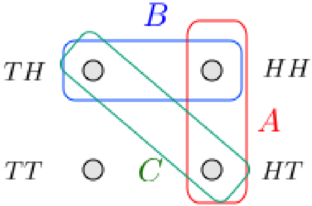

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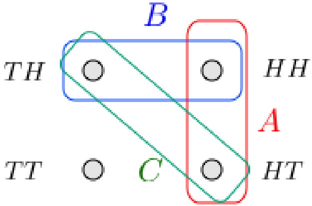

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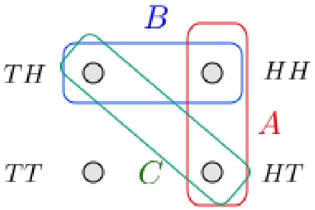

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False: If A did not say anything about C and B did not say anything about C, then $A \cap B$ would not say anything about C.

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Example: Flip a fair coin forever. Let A_n = 'coin n is H.' Then the events A_n are mutually independent.

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