CS70.

1. Random Variables: Brief Review
2. Joint Distributions.
3. Linearity of Expectation

Random Variables: Definitions

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Thus,

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Also,

$$\sum_{a} a \times Pr[X = a] = 3 \times \frac{1}{8} + 2 \times \frac{3}{8} + 1 \times \frac{3}{8} + 0 \times \frac{1}{8}.$$

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$$X_1(\omega) = \begin{cases} 1, & \text{if coin 1 is heads} \\ 0, & \text{otherwise} \end{cases} \qquad X_2(\omega) = \begin{cases} 1, & \text{if coin 2 is heads} \\ 0, & \text{otherwise} \end{cases}$$

Multiple Random Variables: setup.

Joint Distribution: $\{(a, b, Pr[X = a, Y = b]) : a \in \mathscr{A}, b \in \mathscr{B}\}$, where $\mathscr{A}$ ($\mathscr{B}$) is possible values of X (Y).

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Marginal for X: $Pr[X = a] = \sum_{b \in \mathscr{B}} Pr[X = a, Y = b]$.
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X/Y	1	2	3	X
1	.2	.1	.1	.4
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3	.1	0	.2	.3
Y	.3	.1	.2	

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Conditional Probability: $Pr[X=a|Y=b] = \frac{Pr[X=a,Y=b]}{Pr[Y=b]}$.

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- Example: $X, Y, Z \in \{0, 1\}$ three fair coin flips are mutually independent.

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Follows from $Pr[A\cap B]=Pr[A|B]Pr[B]$ (Product rule.)

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Flip a fair coin five times, X = number of Hs in first three flips, Y = number of Hs in last two flips. X and Y are independent.

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Flip a fair coin five times, X = number of Hs in first three flips, Y = number of Hs in last two flips. X and Y are independent.

Indeed:

$$Pr[X=a, Y=b] = \binom{3}{a}\binom{2}{b}2^{-5}$$

Independence: Examples

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Roll two die. X, Y = number of pips on the two dice. X, Y are independent.

Indeed: $Pr[X = a, Y = b] = \frac{1}{36}, Pr[X = a] = Pr[Y = b] = \frac{1}{6}$.

Example 2
Roll two die. X = total number of pips, Y = number of pips on die 1 minus number on die 2. X and Y are not independent.

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$$\begin{aligned}
E[X+Y] &= \sum_{\omega \in \Omega} (X(\omega) + Y(\omega)) Pr[\omega] \\
&= \sum_{\omega \in \Omega} X(\omega) Pr[\omega] + Y(\omega) Pr[\omega] \\
&= \sum_{\omega \in \Omega} X(\omega) Pr[\omega] + \sum_{\omega \in \Omega} Y(\omega) Pr[\omega] \\
&= E[X] + E[Y]
\end{aligned}$$

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Thus, we will write $X = 1_A$.

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&\quad=a_1\sum_\omega X_1(\omega)Pr[\omega]+\cdots+a_n\sum_\omega X_n(\omega)Pr[\omega]\\
&\quad=a_1E[X_1]+\cdots+a_nE[X_n].
\end{aligned}$$

$\square$

Note: If we had defined $Y = a_1X_1+\cdots+a_nX_n$ has had tried to compute $E[Y] = \sum_y yPr[Y=y]$, we would have been in trouble!

Using Linearity - 1: Pips (dots) on dice

Roll a die n times.

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Note: Computing $\sum_x xPr[X = x]$ directly is not easy!

Using Linearity - 2: Fixed point.

Hand out assignments at random to n students.

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Using Linearity - 3: Binomial Distribution.

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Thus, $E[X + b] = E[X] + b$.

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Experiment: Throw m balls into n bins.

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Review: Harmonic sum

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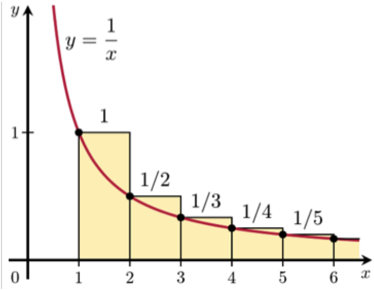

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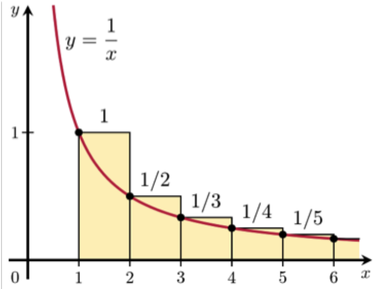

A good approximation is

$H(n) \approx \ln(n) + \gamma$ where $\gamma \approx 0.58$ (Euler-Mascheroni constant).

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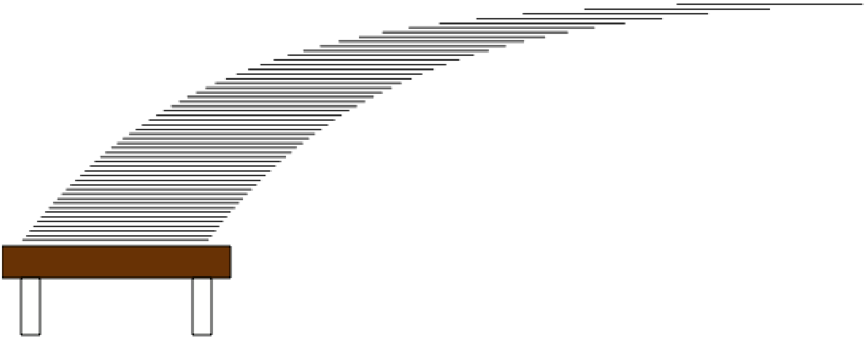

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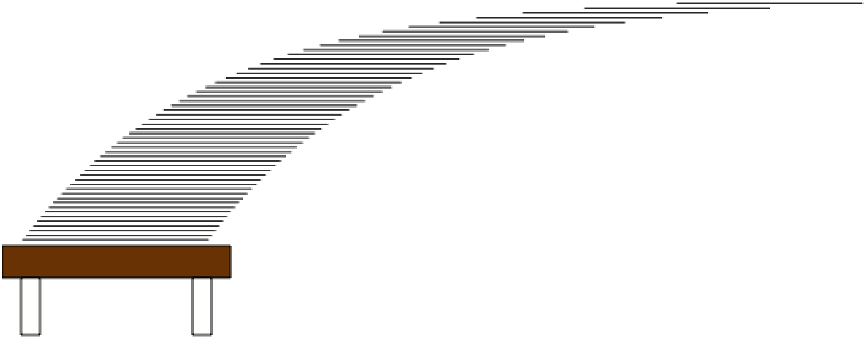

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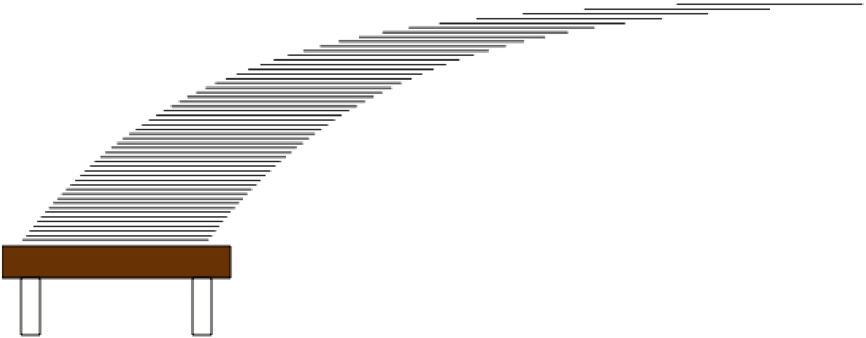


If each card has length 2, the stack can extend $H(n)$ to the right of the table. As n increases, you can go as far as you want!

Paradox

par·a·dox

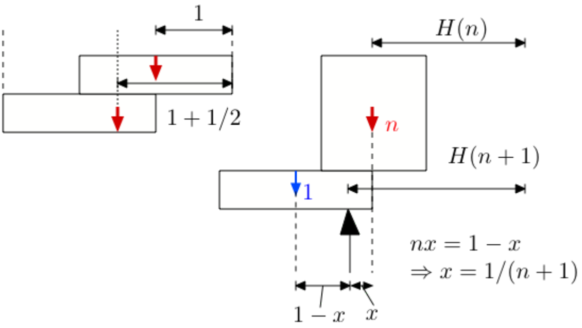
/ˈperəˌdäks/

noun

a statement or proposition that, despite sound (or apparently sound) reasoning from acceptable premises, leads to a conclusion that seems senseless, logically unacceptable, or self-contradictory.
"a potentially serious conflict between quantum mechanics and the general theory of relativity known as the information paradox"

- a seemingly absurd or self-contradictory statement or proposition that when investigated or explained may prove to be well founded or true.
  "in a paradox, he has discovered that stepping back from his job has increased the rewards he gleans from it"
  synonyms: contradiction, contradiction in terms, self-contradiction, inconsistency, incongruity; More
- a situation, person, or thing that combines contradictory features or qualities.
  "the mingling of deciduous trees with elements of desert flora forms a fascinating ecological paradox"

Stacking



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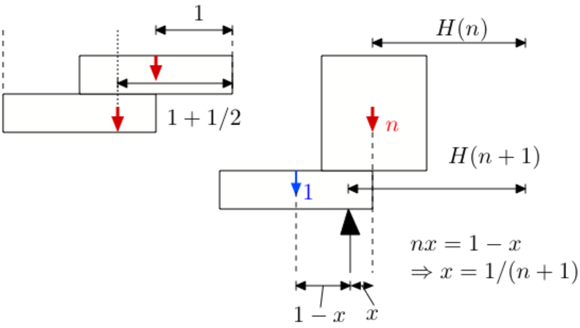

The cards have width 2.

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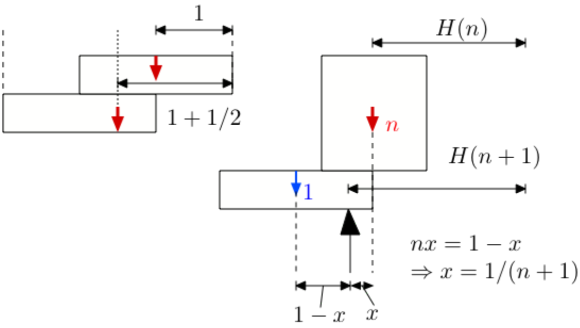

The cards have width 2. Induction shows that the center of gravity after n cards is $H(n)$ away from the right-most edge.
Video.

Calculating $E[g(X)]$

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$\square$

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Thus,

$$E[Y] = 4\frac{2}{6} + 1\frac{2}{6} + 0\frac{1}{6} + 9\frac{1}{6} = \frac{19}{6}.$$

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