CS70

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Variance/ Covariance.

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Model with binomial.

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For (1) we used $m \ll n$; for (2) we used $(1 - a/n)^n \approx e^{-a}$.

Poisson Distribution: Definition and Mean

Definition Poisson Distribution with parameter $\lambda > 0$

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$\square$

Simeon Poisson

The Poisson distribution is named after:

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Siméon Denis Poisson (1781–1840)

Variance

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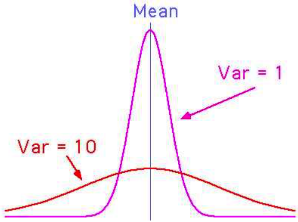
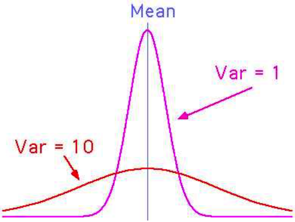


Variance



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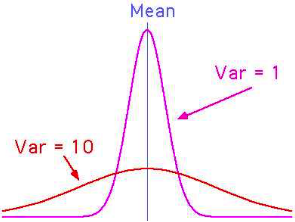
Variance



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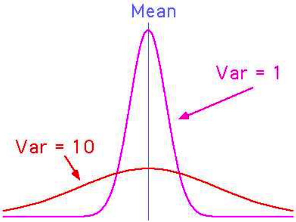

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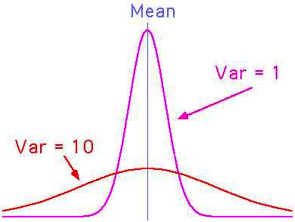

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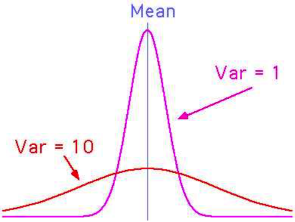

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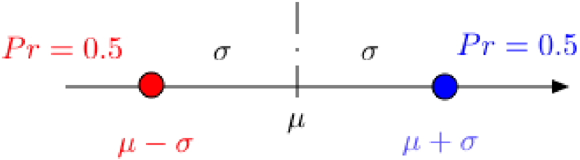

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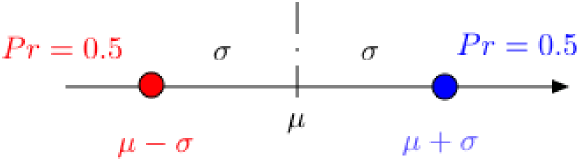

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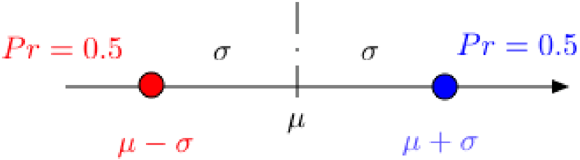

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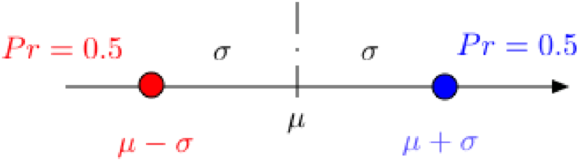

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$$var(X) = \sigma^2 \text{ and } \sigma(X) = \sigma.$$

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Exercise: How big can you make $\frac{\sigma(X)}{E[|X - E[X]|]}$?

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Uniform

Assume that $Pr[X=i]=1/n$ for $i \in \{1,\ldots,n\}$. Then

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This gives

$$
var(X) = \frac{1+3n+2n^2}{6} - \frac{(n+1)^2}{4} = \frac{n^2-1}{12}.
$$

(Sort of $\int_0^{1/2} x^2 dx = \frac{x^3}{3}$.)

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$Var(X) = E(X^2) - (E(X))^2 = 2 - 1 = 1.$

Variance: binomial.

$$E[X^2] \quad = \quad \sum_{i=0}^{n} i^2 \binom{n}{i} p^i (1-p)^{n-i}.$$

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Let's do something else.
Maybe not much easier...but there is a payoff.

Properties of variance.

1. $Var(cX) = c^2 Var(X)$, where c is a constant.

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&= E(X^2)+E(Y^2)+E(Z^2)+\cdots+0+\cdots+0\\
&= var(X)+var(Y)+var(Z)+\cdots.
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$\square$

Variance of Binomial Distribution.

Flip coin with heads probability p.

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Poisson Distribution: Variance.

Definition Poisson Distribution with parameter $\lambda > 0$

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$$\begin{aligned} E[(X-E[X])(Y-E[Y])] &= E[XY - E[X]Y - XE[Y] + E[X]E[Y]] \\ &= E[XY] - E[X]E[Y] - E[X]E[Y] + E[X]E[Y] \\ &= E[XY] - E[X]E[Y]. \end{aligned}$$

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Theorem: $-1 \leq corr(X, Y) \leq 1$.
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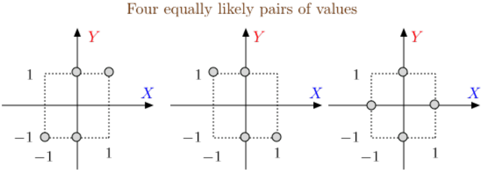
$\rightarrow E[XY] \geq -1$.

$\square$

Shifting and scaling doesn't change correlation.

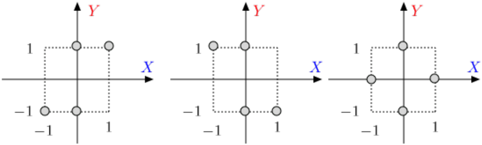
Examples of Covariance



$cov(X, Y) = 1/2$ $\quad$ $cov(X, Y) = -1/2$ $\quad$ $cov(X, Y) = 0$

Examples of Covariance

Four equally likely pairs of values



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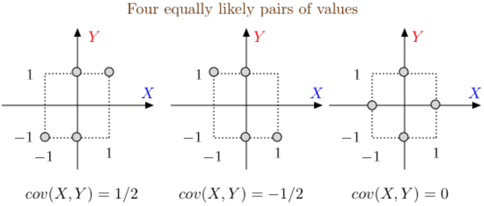

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When $cov(X, Y) > 0$, the RVs X and Y tend to be large or small together.

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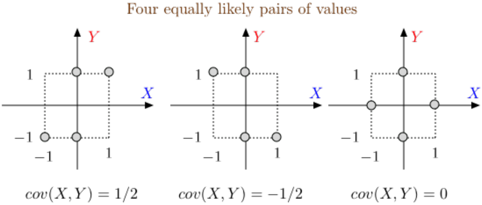

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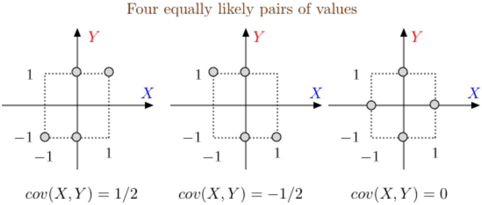

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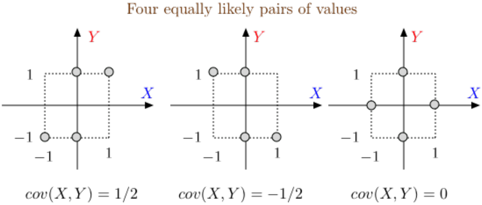

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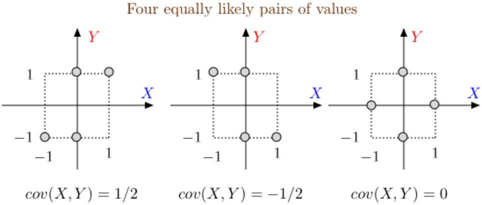

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When $cov(X, Y) = 0$, we say that X and Y are uncorrelated.

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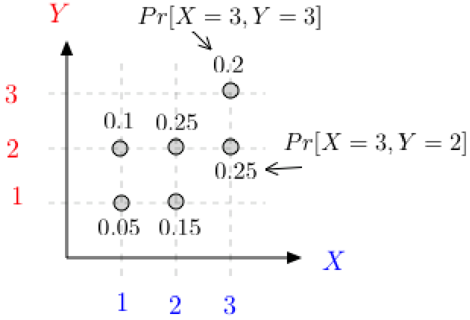

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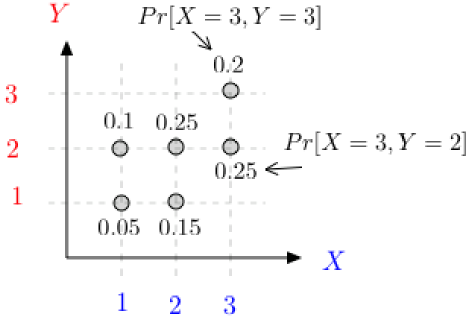

$$E[X] = 1 \times 0.15 + 2 \times 0.4 + 3 \times 0.45 = 2.3$$

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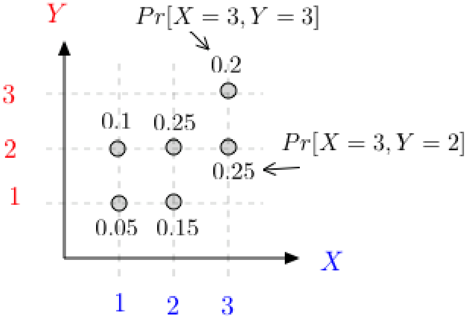

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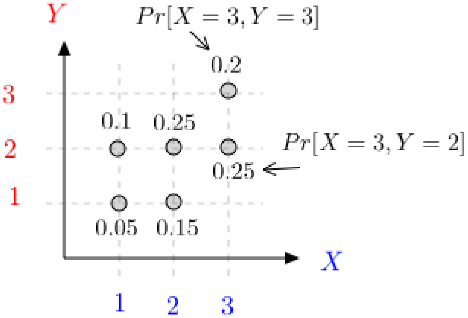

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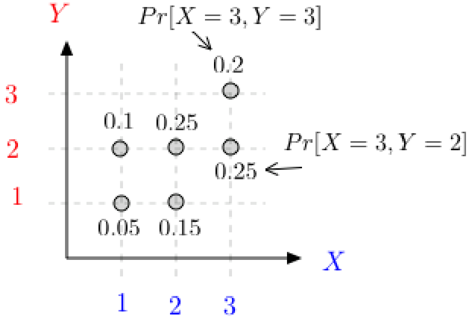

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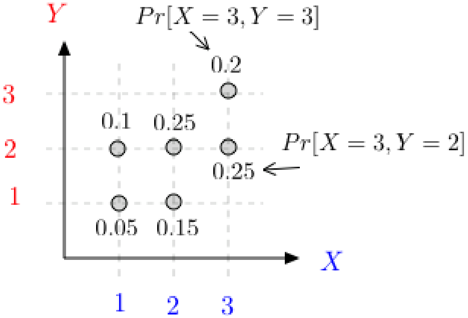

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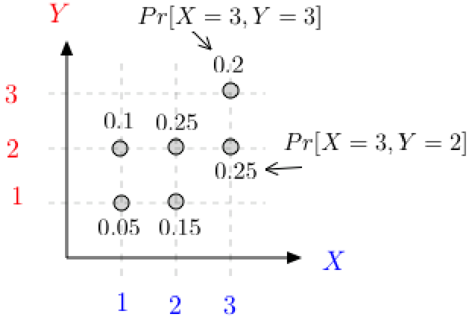

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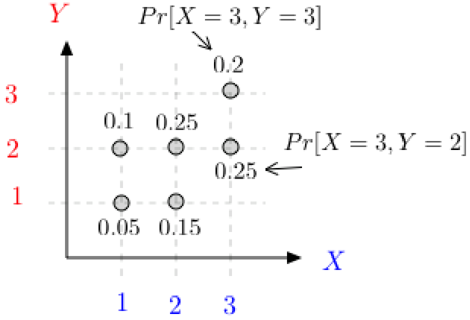

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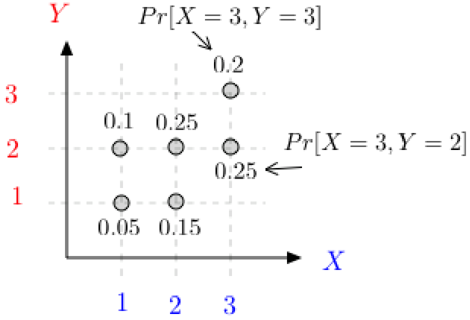

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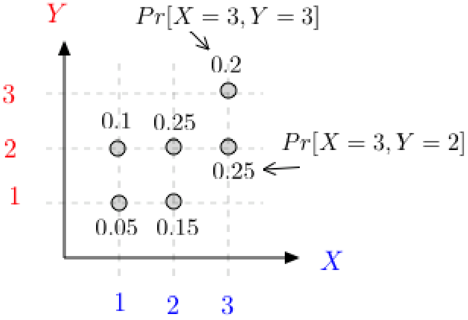

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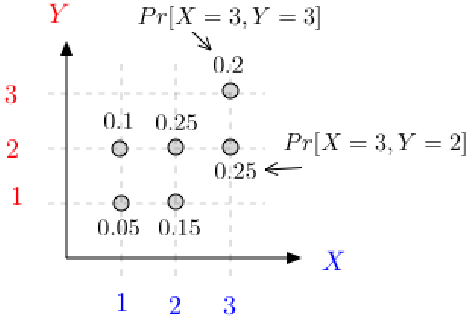

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Fact

(a) $var[X] = cov(X,X)$

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Fact
(a) $var[X] = cov(X, X)$
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Proof:

(a)-(b)-(c) are obvious.

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