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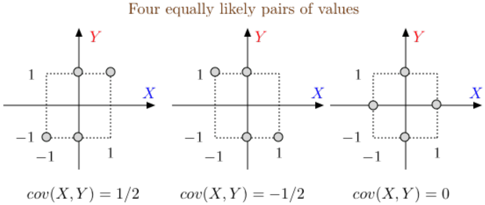
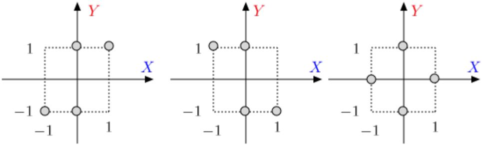
$r^2 = corr(X,Y)^2$ is fraction of variance of Y explained by X.

Examples of Covariance



Examples of Covariance

Four equally likely pairs of values



$cov(X, Y) = 1/2$ $cov(X, Y) = -1/2$ $cov(X, Y) = 0$

Note that $E[X] = 0$ and $E[Y] = 0$ in these examples. Then $cov(X, Y) = E[XY]$.

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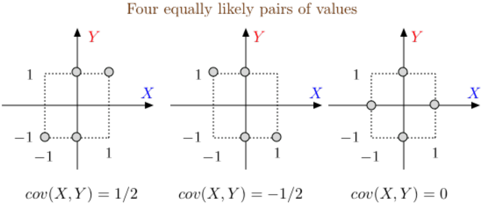

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When $cov(X, Y) > 0$, the RVs X and Y tend to be large or small together.

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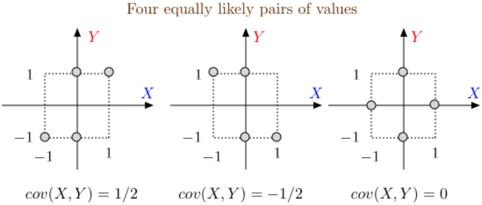

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When $cov(X, Y) > 0$, the RVs X and Y tend to be large or small together. X and Y are said to be positively correlated.

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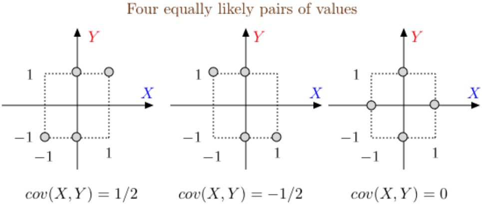

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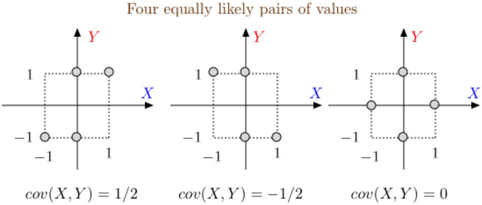

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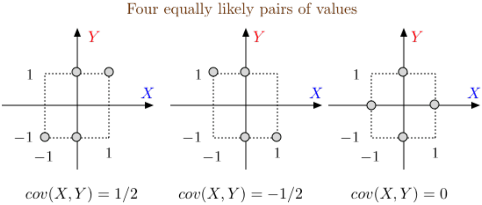

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When $cov(X, Y) = 0$, we say that X and Y are uncorrelated.

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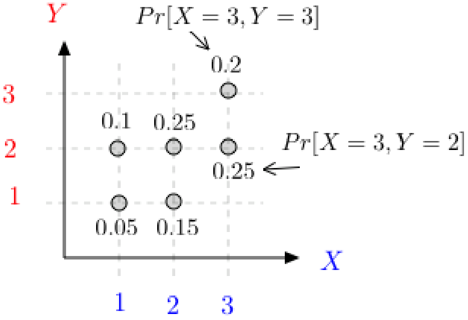

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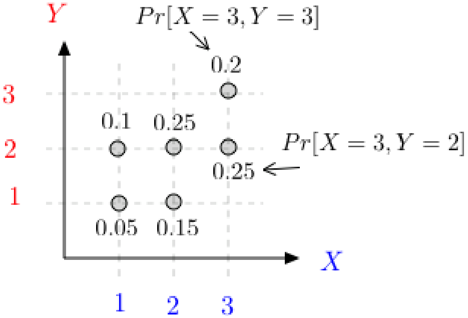

$$E[X] = 1 \times 0.15 + 2 \times 0.4 + 3 \times 0.45 = 2.3$$

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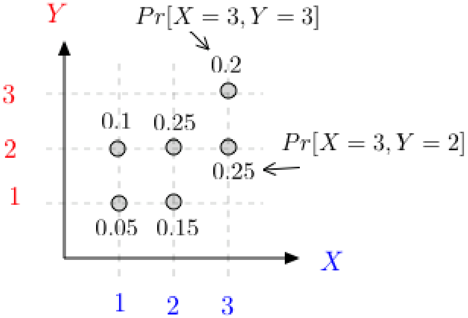

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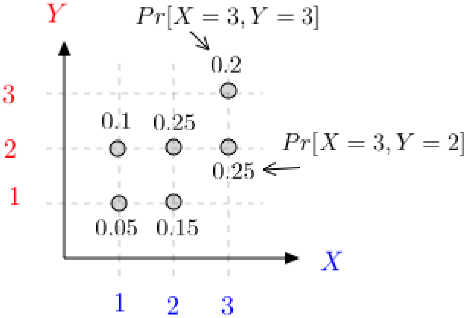

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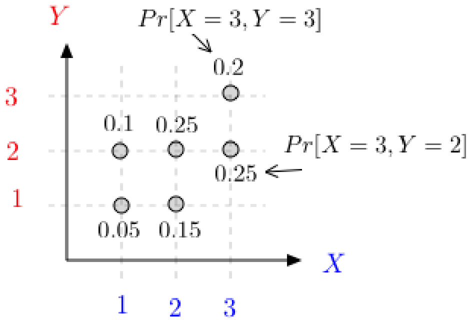

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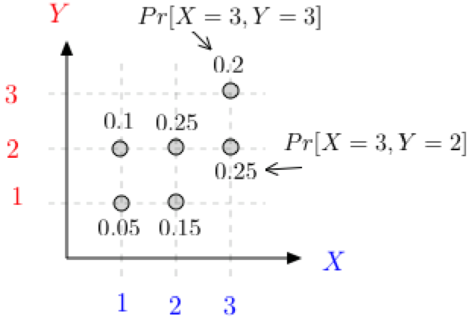
Examples of Covariance



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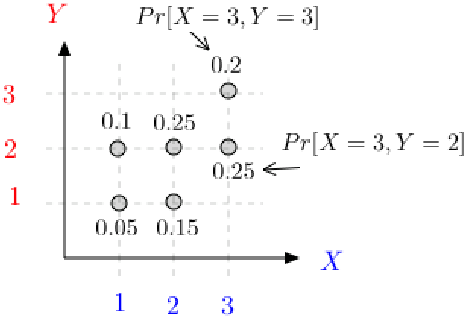

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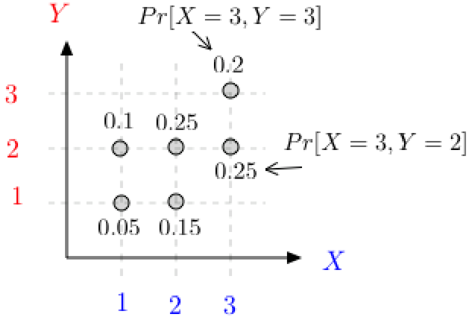

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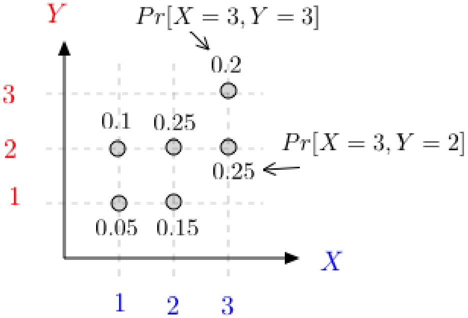

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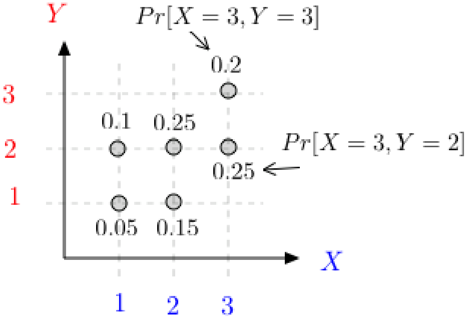

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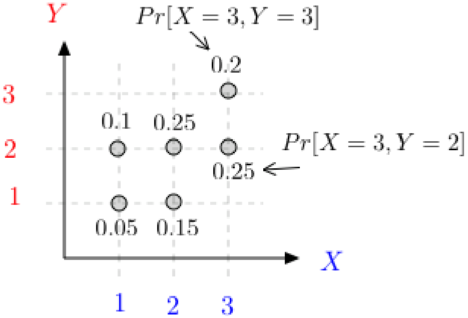

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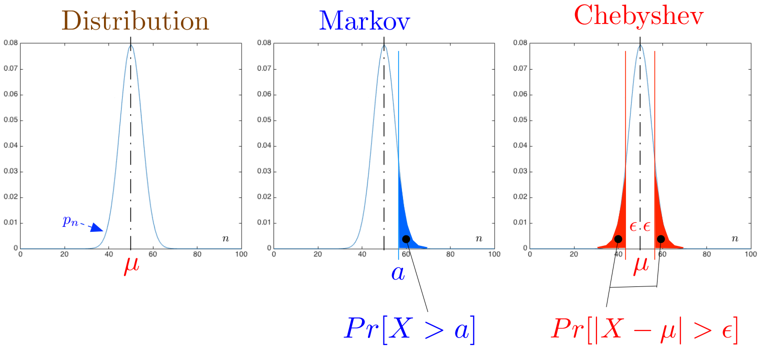
Inequalities: An Overview



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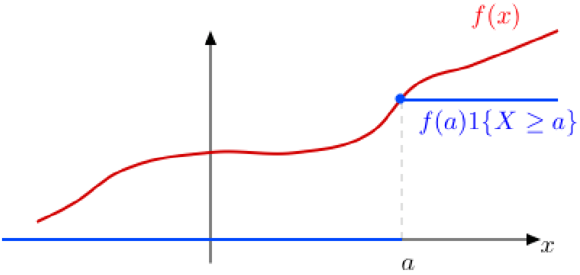
That is, $\sum_v Pr[X = v] 1\{v \geq a\} \leq \sum_v Pr[X = v] \frac{f(v)}{f(a)}$.

$\square$

A picture



$$f(a)1\{X \geq a\} \leq f(x) \Rightarrow 1\{X \geq a\} \leq \frac{f(X)}{f(a)}$$

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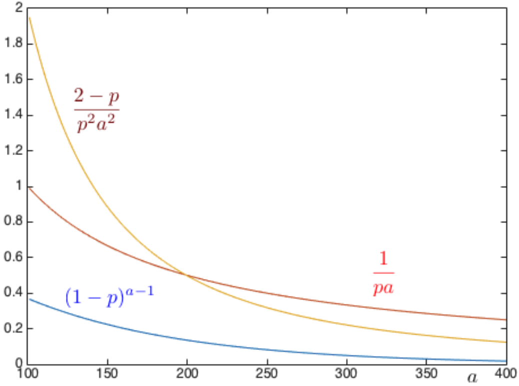

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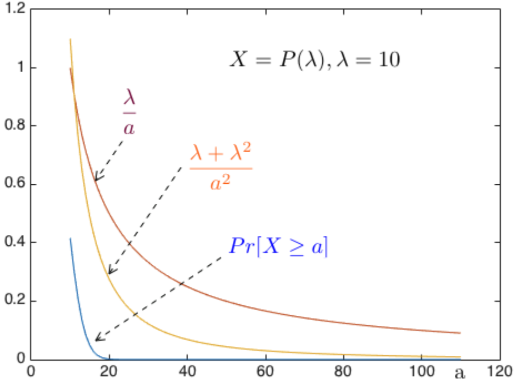

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This result confirms that the variance measures the "deviations from the mean."

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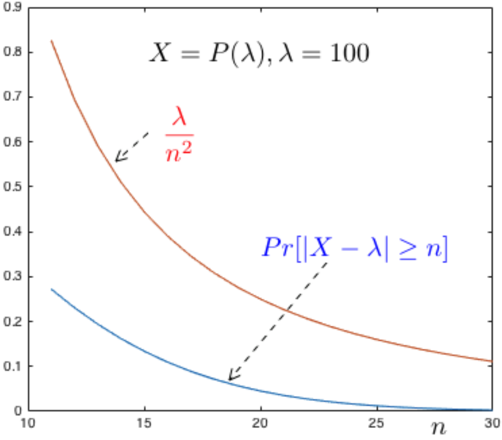

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Also, if $a > \lambda$, then $X \geq a \Rightarrow X - \lambda \geq a - \lambda > 0$

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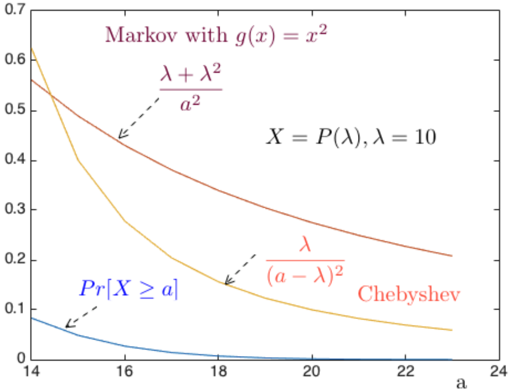

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We look at a general case next.

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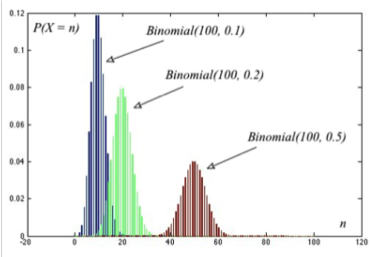

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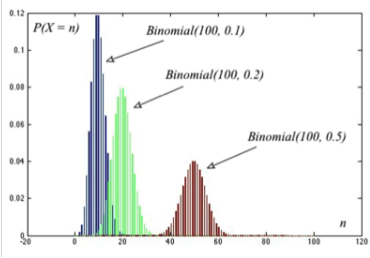

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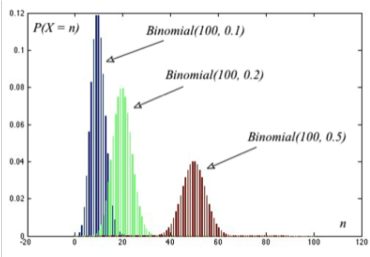

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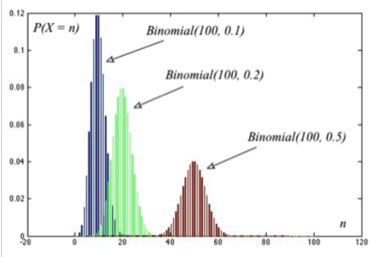

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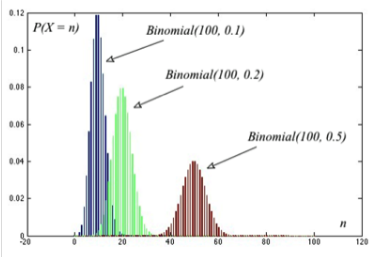

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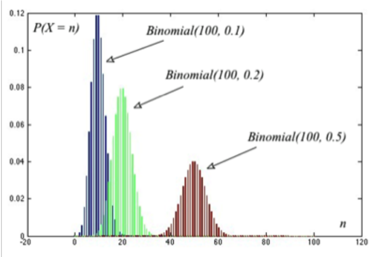

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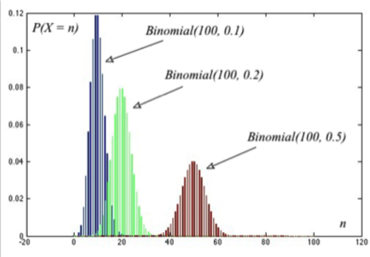

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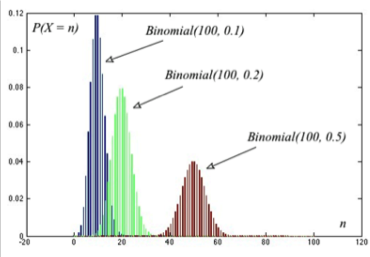

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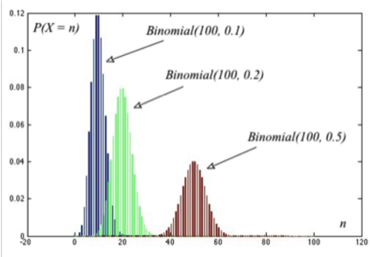

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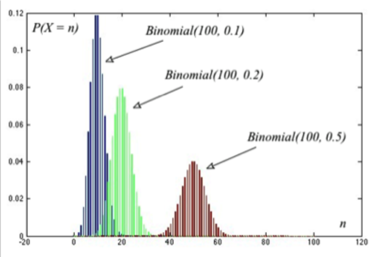

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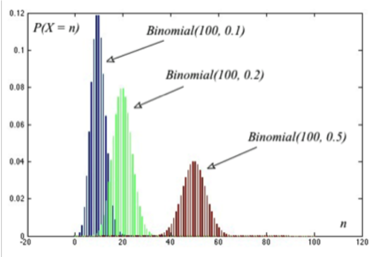

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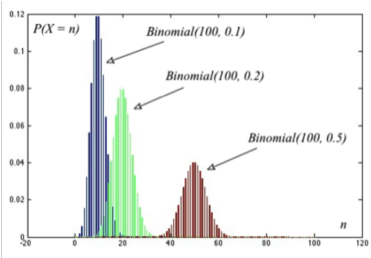

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Theorem:
Let X_n be i.i.d. with mean μ and variance σ^2.
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Hence,

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$\square$

Confidence interval for p in $B(p)$

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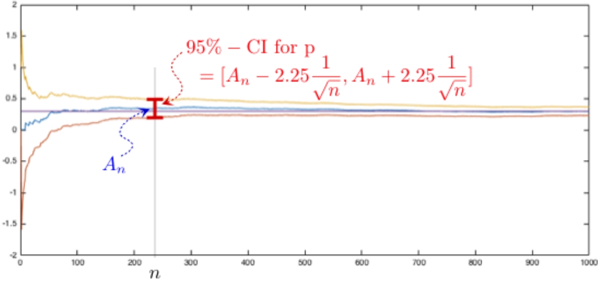

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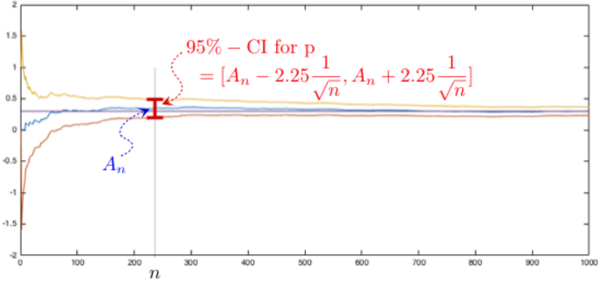

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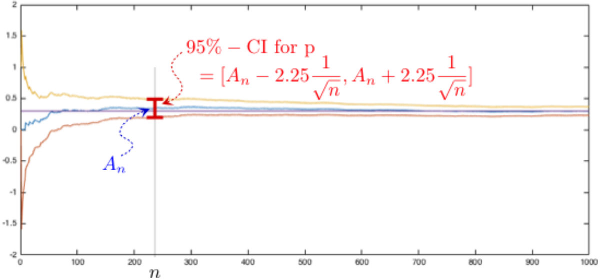

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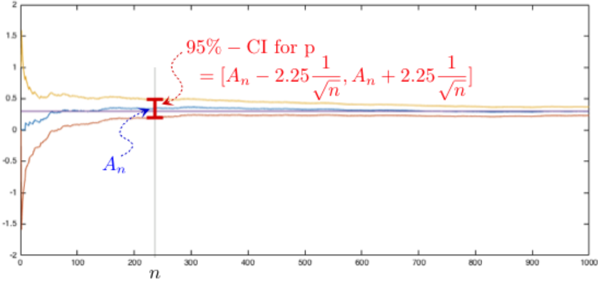

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Confidence interval for p in $B(p)$

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Improved CI:

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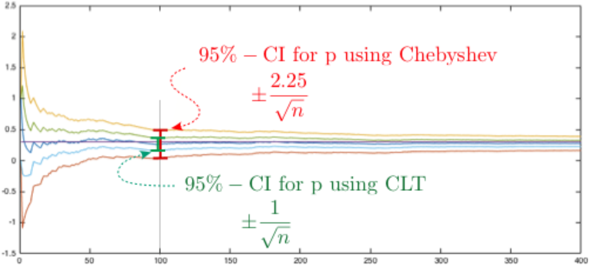
Improved CI: In fact, one can replace 2.25 by 1.



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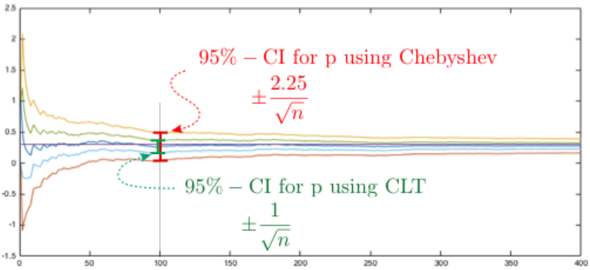

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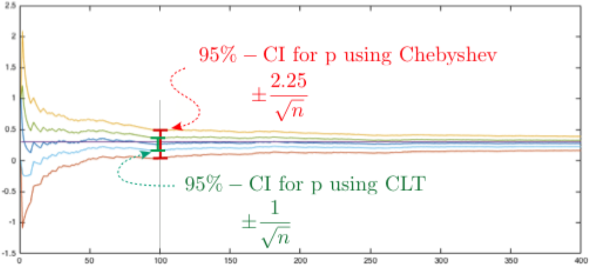

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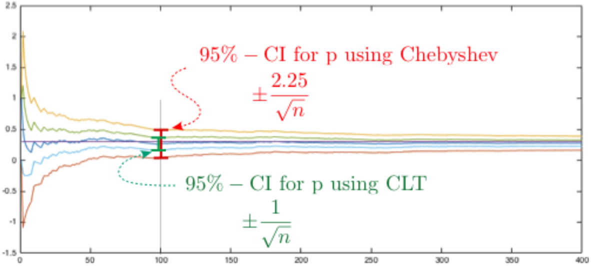

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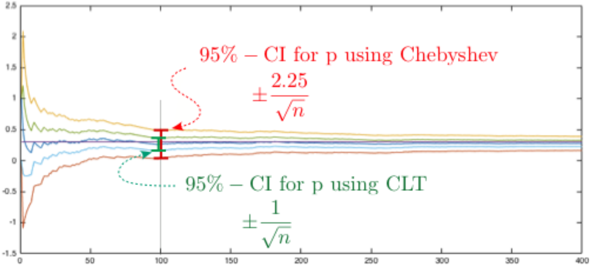

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Examples:

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Examples: $[0.7A_{100}, 1.8A_{100}]$

Confidence Interval for $1/p$ in $G(p)$

Let X_n be i.i.d. $G(p)$. Define $A_n = (X_1 + \cdots + X_n)/n$.

Theorem:

$$[\frac{A_n}{1+4.5/\sqrt{n}}, \frac{A_n}{1-4.5/\sqrt{n}}] \text{ is a 95\%-CI for } \frac{1}{p}.$$

Proof: We know that

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Example: With $n = 100$ and $A_n - B_n = 0.2$, $Pr[p_A > p_B] \geq 1 - \frac{1}{8} = 0.875$.

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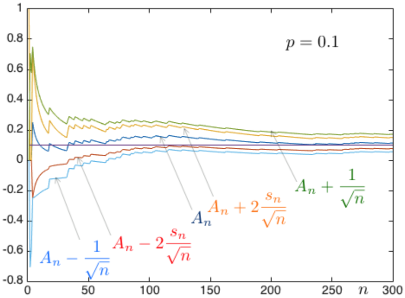

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