CS70: Markov Chains.

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1. Examples
2. Definition
3. First Passage Time

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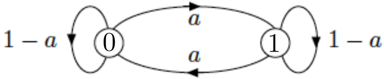

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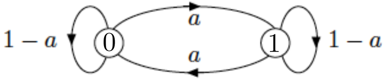

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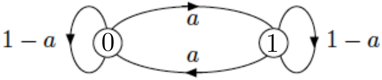

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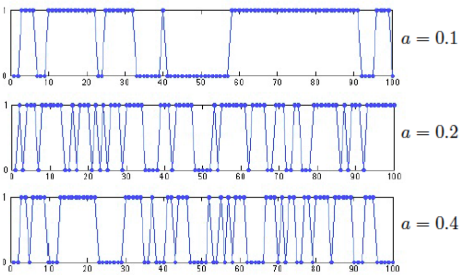

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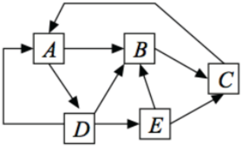
At each step, the MC follows one of the outgoing arrows of the current state, with equal probabilities.



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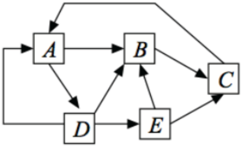

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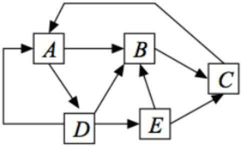

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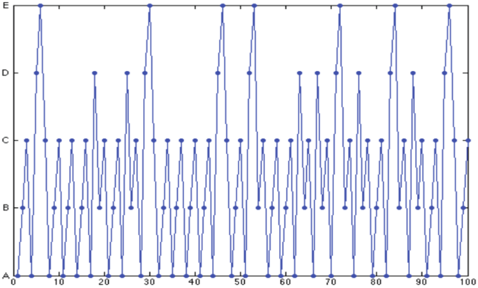

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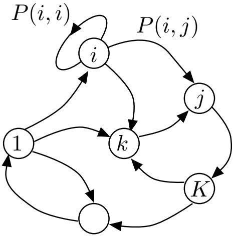

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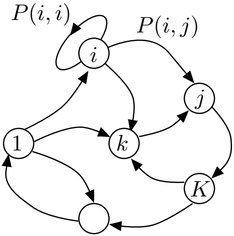

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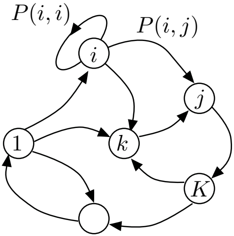

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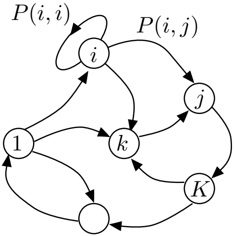

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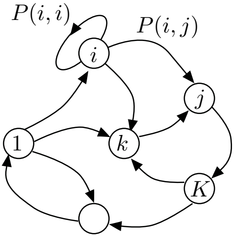

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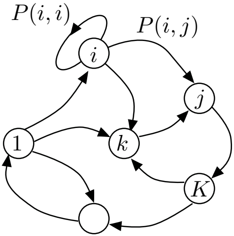

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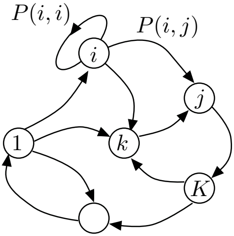

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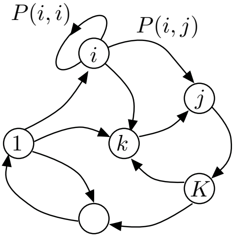

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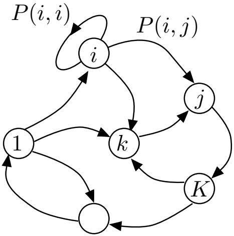

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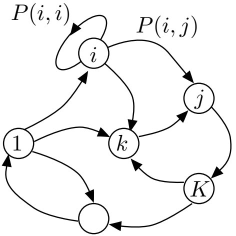

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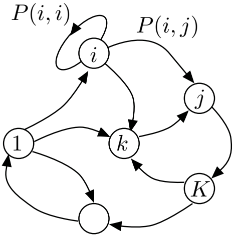

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  $Pr[X_{n+1} = j \mid X_0, \ldots, X_n = i] = P(i,j), i,j \in \mathscr{X}$.

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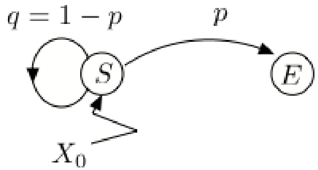

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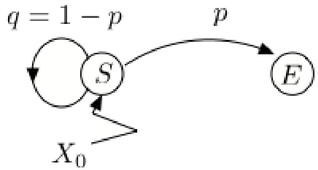

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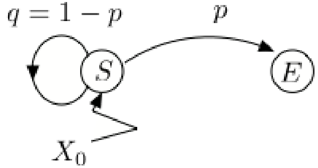

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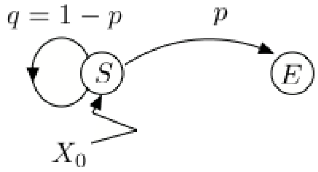

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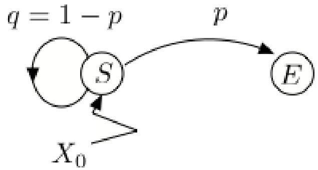

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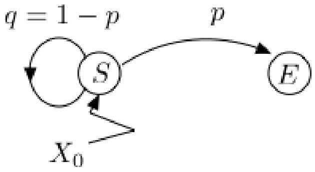

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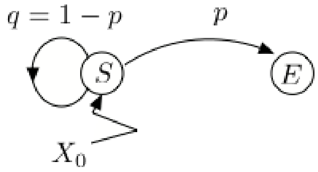

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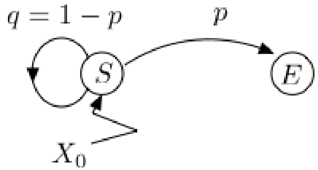

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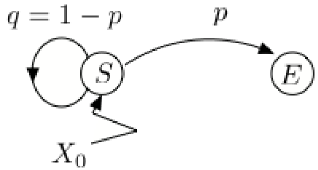

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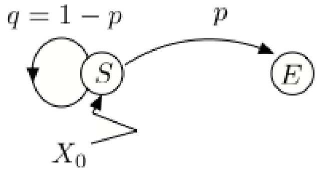

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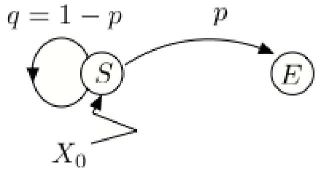

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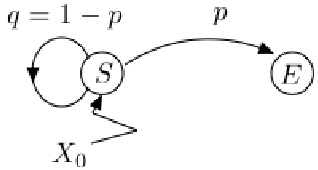

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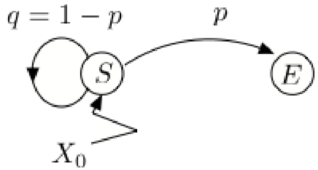

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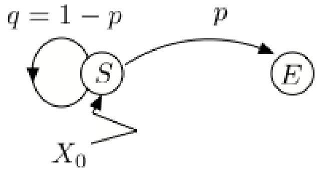

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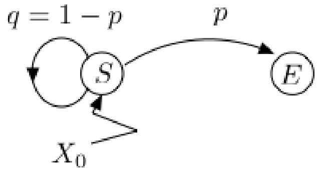

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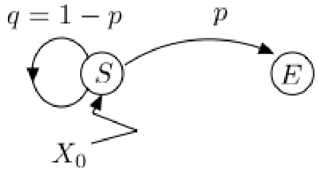

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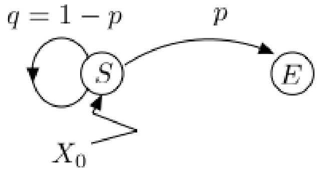

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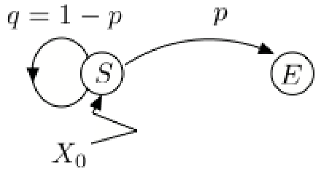

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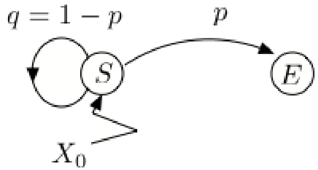

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$$N = 1 + (1 - Z) \times N' + Z \times 0.$$

Z and N' are independent. Also, $E[N'] = E[N] = \beta(S)$.
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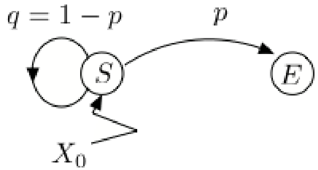
Let's flip a coin with $Pr[H] = p$ until we get H. How many flips, on average?



Let $\beta(S)$ be the average time until E.
Then,

$$\beta(S) = 1 + q\beta(S) + p0.$$

Justification: N – number of steps until E, starting from S.
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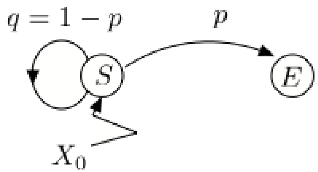

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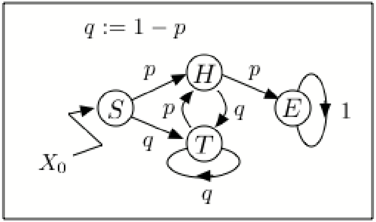

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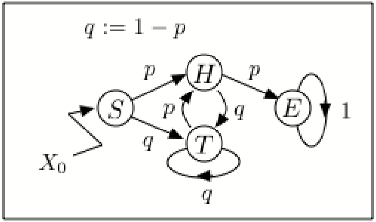

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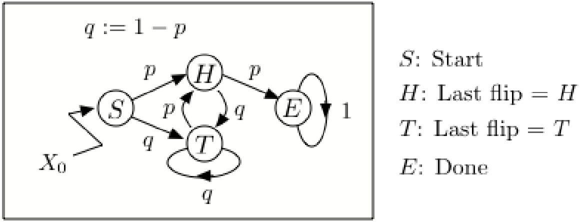

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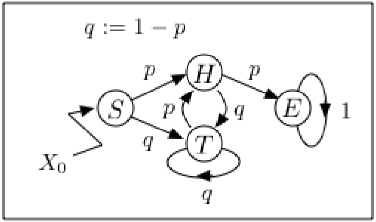

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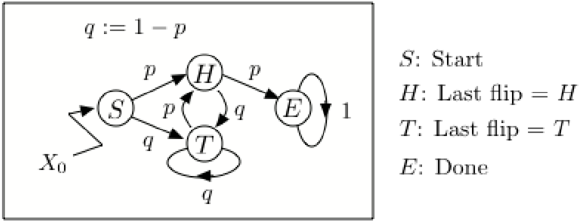

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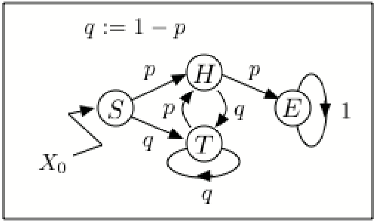

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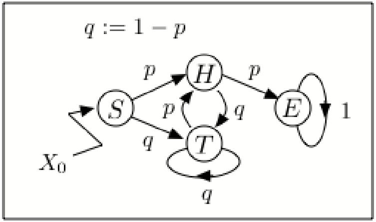

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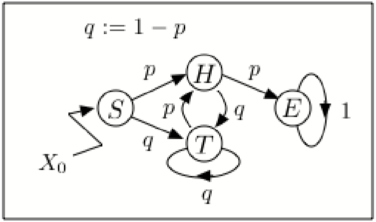

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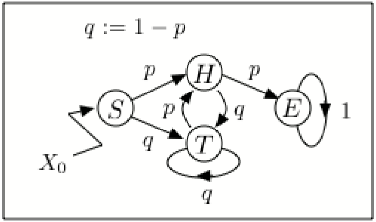

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Solving, we find $\beta(S) = 2 + 3qp^{-1} + q^2p^{-2}$.

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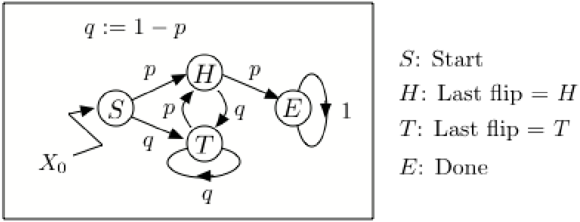

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Solving, we find $\beta(S) = 2 + 3qp^{-1} + q^2p^{-2}$. (E.g., $\beta(S) = 6$ if $p = 1/2$.)

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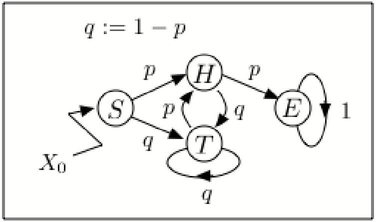

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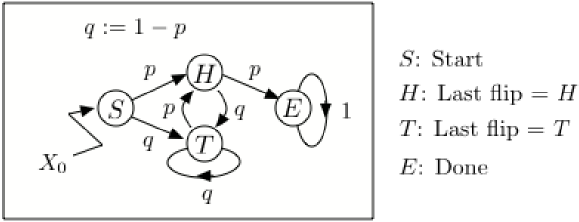
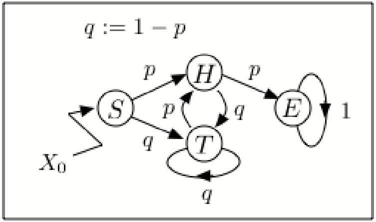


Let us justify the first step equation for $\beta(T)$.

First Passage Time - Example 2



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Let us justify the first step equation for $\beta(T)$. The others are similar.

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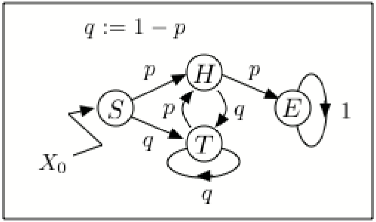

S: Start

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Let us justify the first step equation for $\beta(T)$. The others are similar.

$N(T)$ – number of steps, starting from T until the MC hits E.

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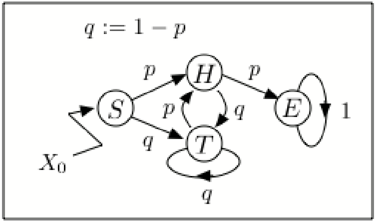

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Let us justify the first step equation for $\beta(T)$. The others are similar.

$N(T)$ – number of steps, starting from T until the MC hits E.
$N(H)$ – be defined similarly.

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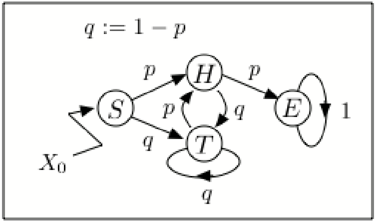

S: Start

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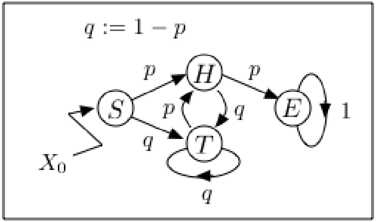

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$$N(T) = 1 + Z \times N(H) + (1 - Z) \times N'(T)$$

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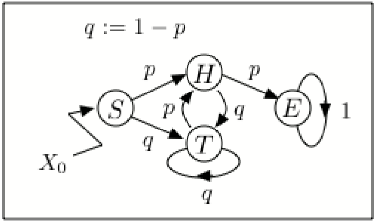

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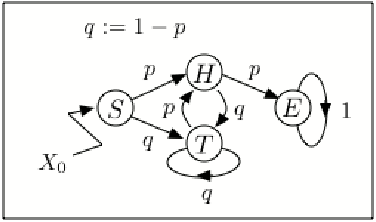

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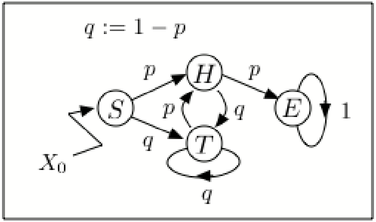

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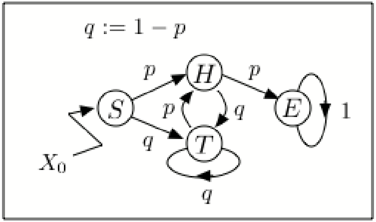

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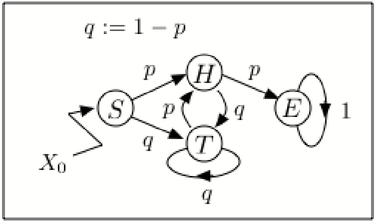

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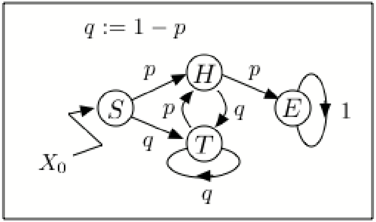

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where $Z = 1\{\text{first flip in } T \text{ is } H\}$. Since Z and $N(H)$ are independent, and Z and $N'(T)$ are independent, taking expectations, we get

$$E[N(T)] = 1 + pE[N(H)] + qE[N'(T)],$$

i.e.,

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First Passage Time - Example 3

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You roll a balanced six-sided die until the sum of the last two rolls is 8.

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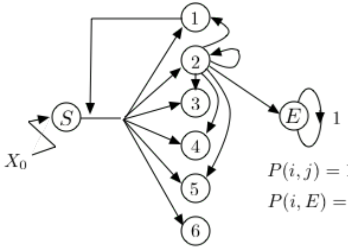


S = Start; E = Done

i = Last roll is i, not done

$P(S, j) = 1/6, j = 1, \ldots, 6$

$P(1, j) = 1/6, j = 1, \ldots, 6$

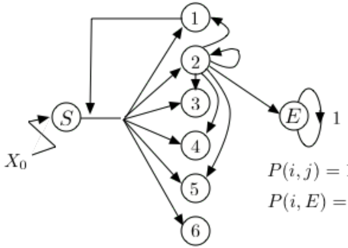
$P(i, j) = 1/6, i = 2, \ldots, 6; 8 - i \neq j \in \{1, \ldots, 6\}$

$P(i, E) = 1/6, i = 2, \ldots, 6$

The arrows out of $3, \ldots, 6$ (not shown) are similar to those out of 2.

First Passage Time - Example 3

You roll a balanced six-sided die until the sum of the last two rolls is 8. How many times do you have to roll the die, on average?



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$P(i, E) = 1/6, i = 2, \ldots, 6$

The arrows out of $3, \ldots, 6$ (not shown) are similar to those out of 2.

$$\beta(S) = 1 + \frac{1}{6} \sum_{j=1}^{6} \beta(j);$$

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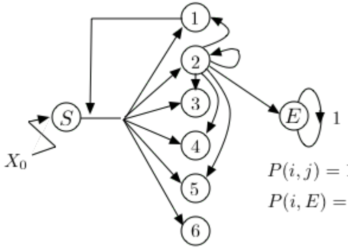

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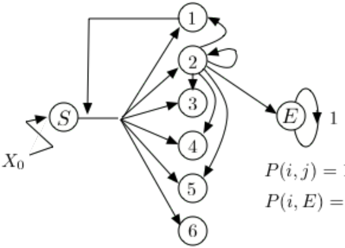

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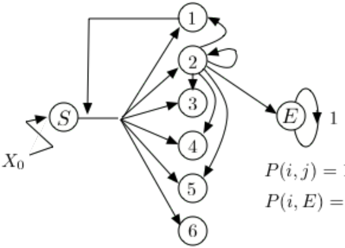

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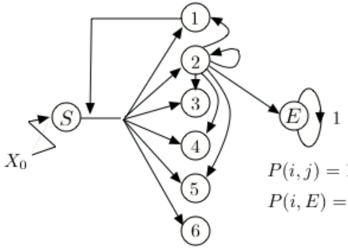

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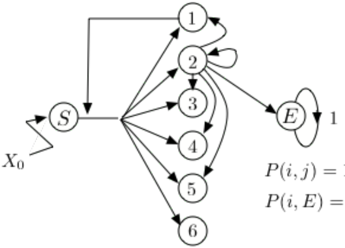

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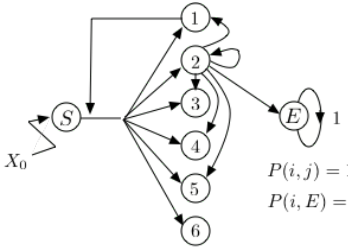

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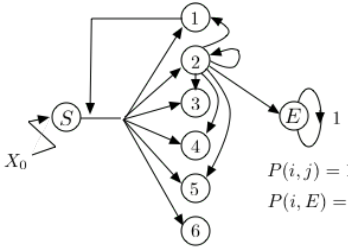

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$$\Rightarrow \cdots \beta(S) = 8.4.$$

First Passage Time - A before B

Game of “heads or tails” using coin with ‘heads’ probability $p < 0.5$.

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First Passage Time - A before B

- Game of “heads or tails” using coin with ‘heads’ probability $p < 0.5$.
- Start with \$10.
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- What is the probability that you reach \$100 before \$0?

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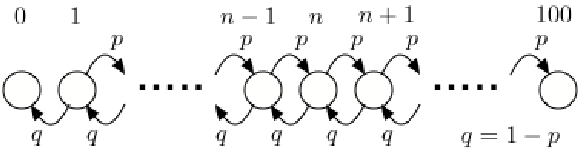

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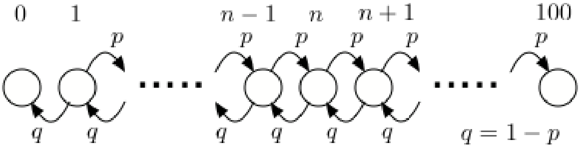

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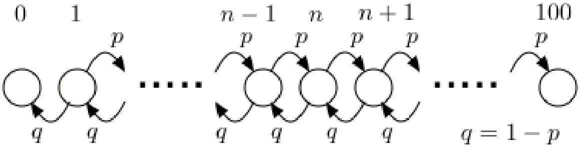

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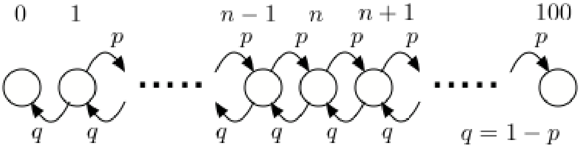

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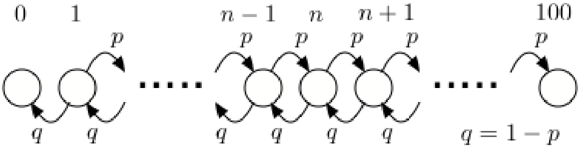

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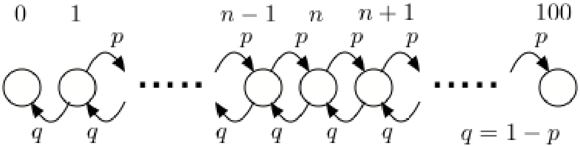

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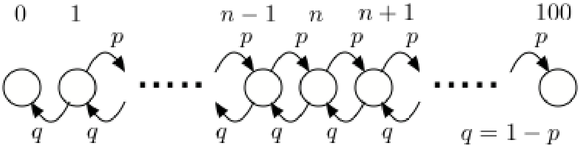

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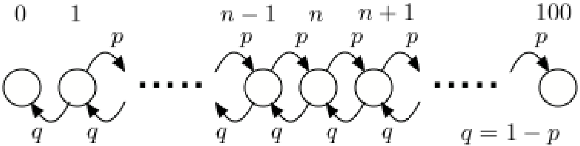

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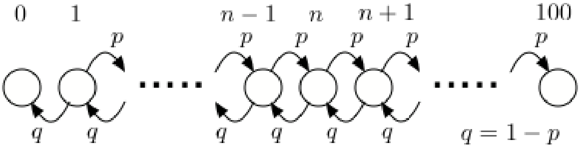

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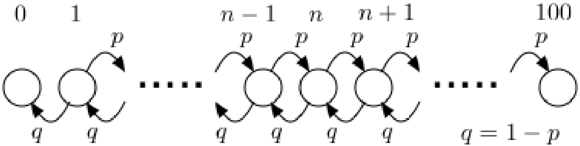

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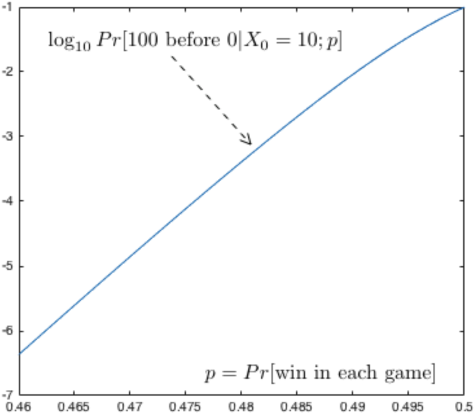

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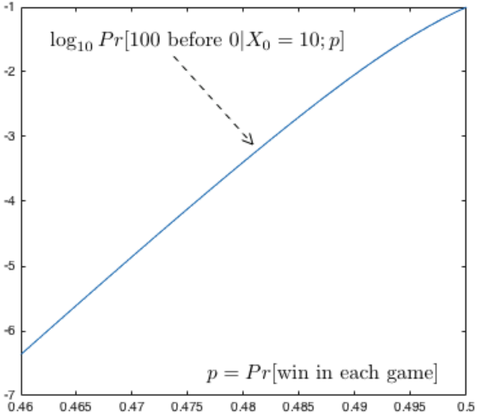

Less than 1 in a 1000.

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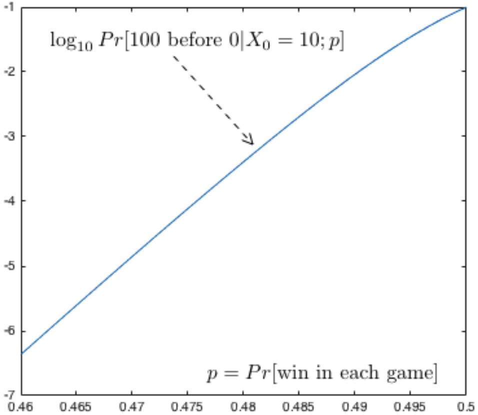

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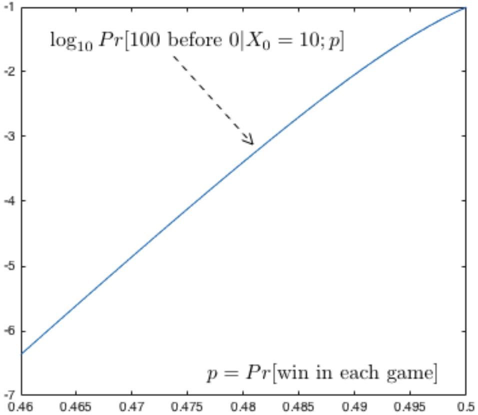

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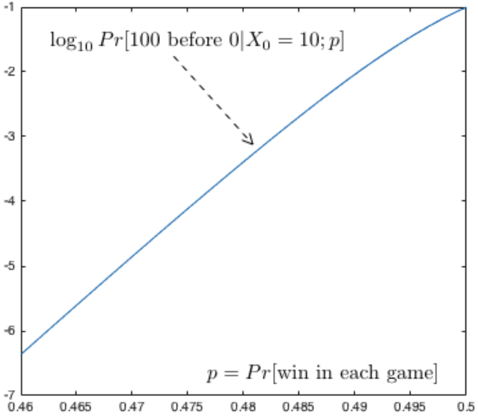

Less than 1 in a 1000. Morale of example: Money in Vegas stays in Vegas.

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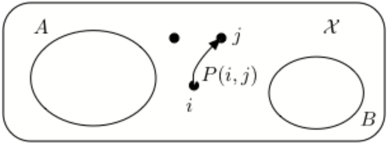

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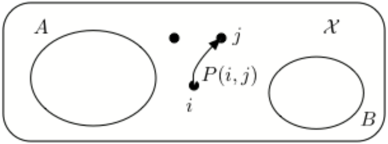

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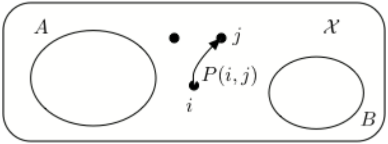

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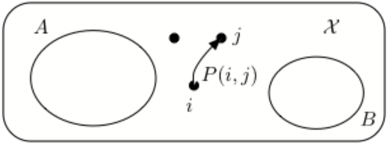

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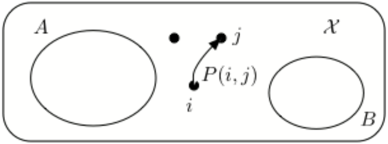

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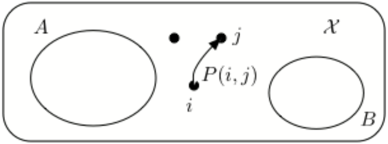

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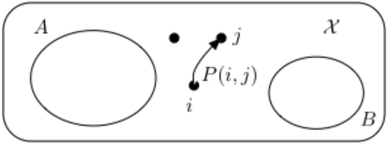

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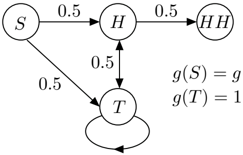

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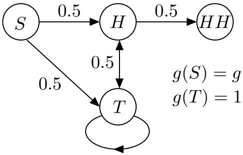

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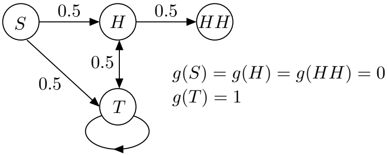

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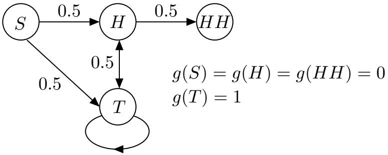

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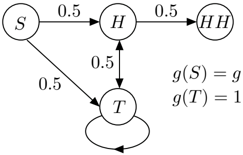

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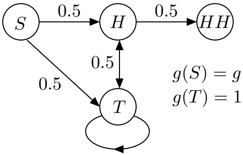

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Solving, we find $\gamma(S) = 2.5$.

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  - Finite set $\mathscr{X}$; π_0; $P = \{P(i,j), i,j \in \mathscr{X}\}$;
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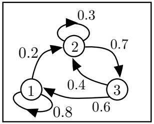
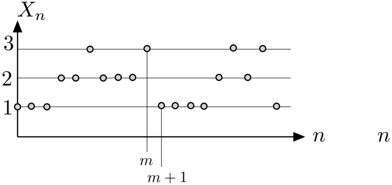
Distribution of X_n





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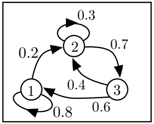

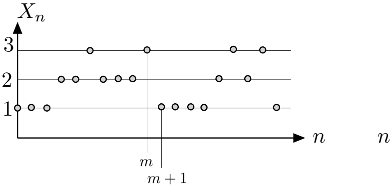

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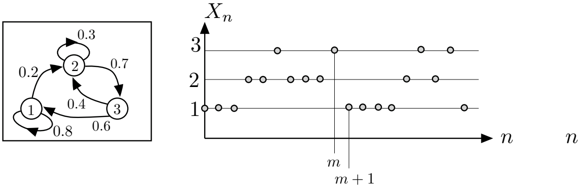

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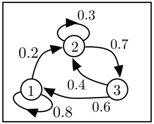

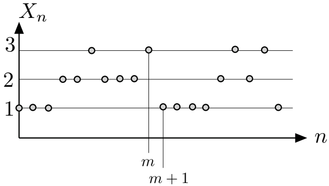

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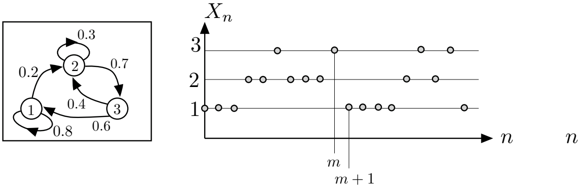

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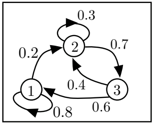

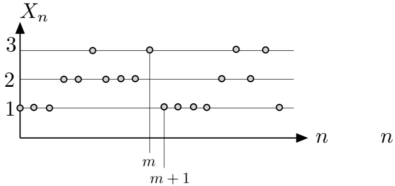

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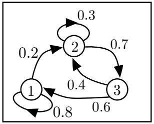

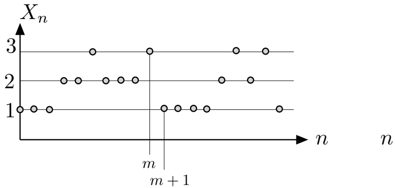

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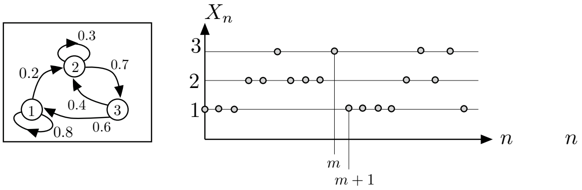

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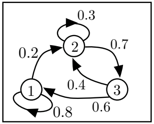

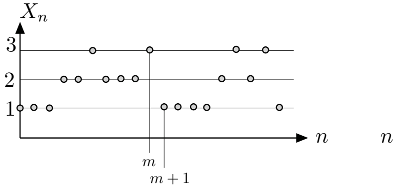

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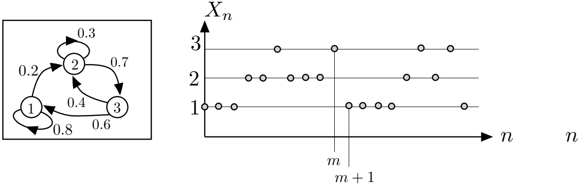

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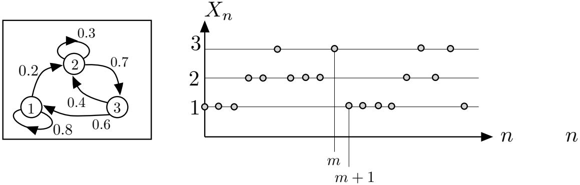

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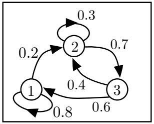

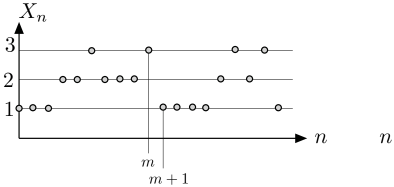

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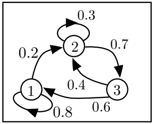

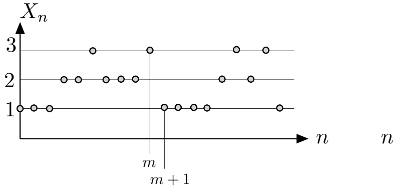

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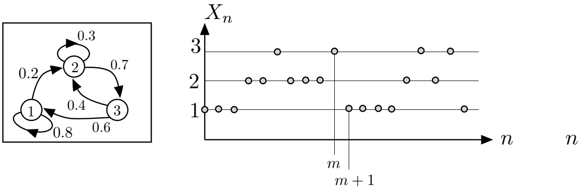

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Sometimes the distribution as $n \to \infty$

Stationary: Example

Example 1:

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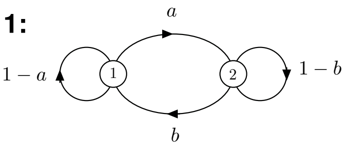

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

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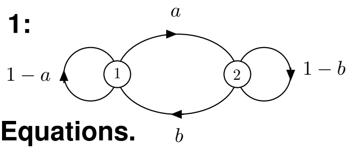

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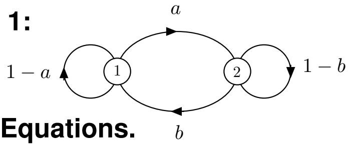

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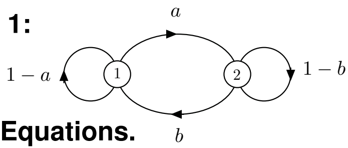

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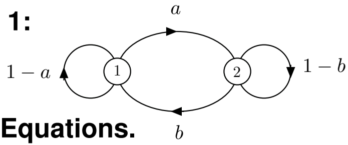

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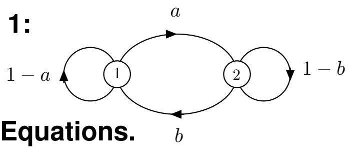

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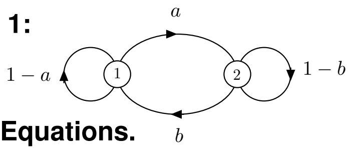

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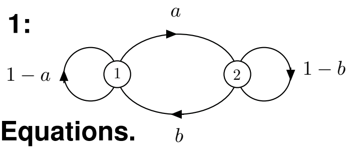

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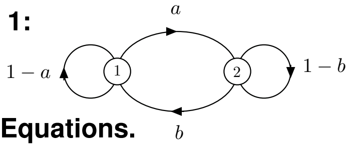

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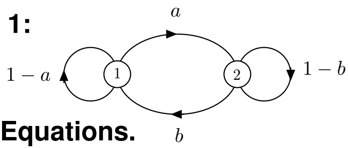

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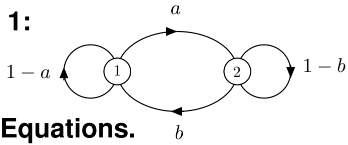

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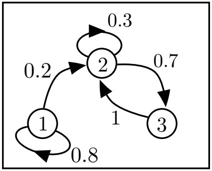

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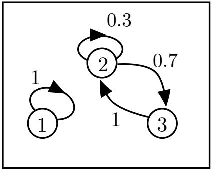

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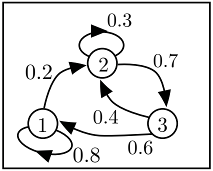

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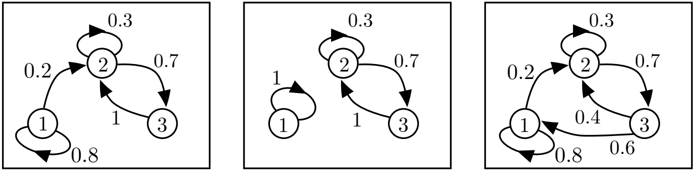

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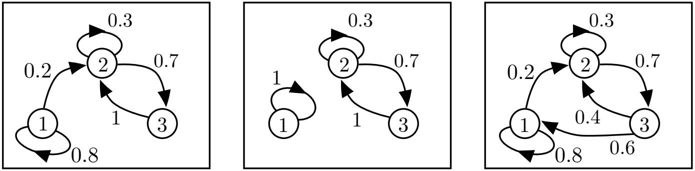

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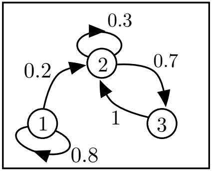

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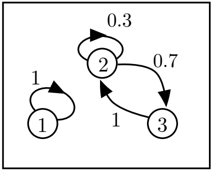

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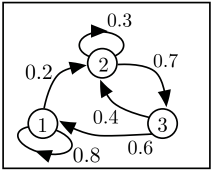

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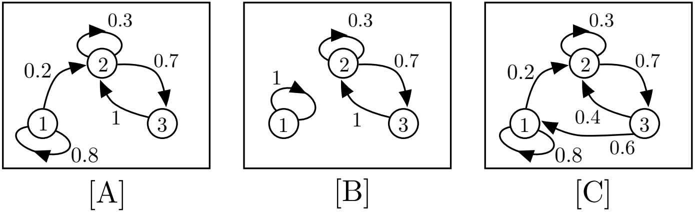

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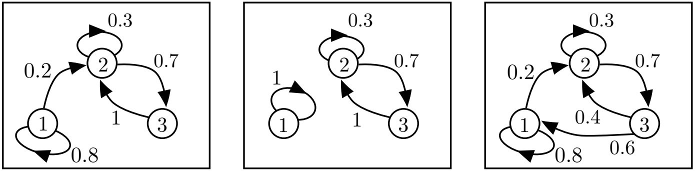

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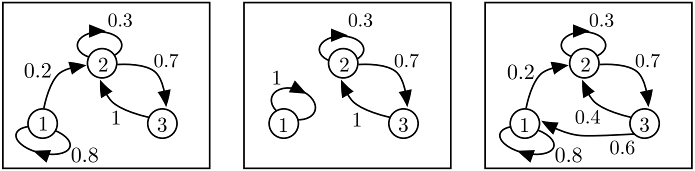

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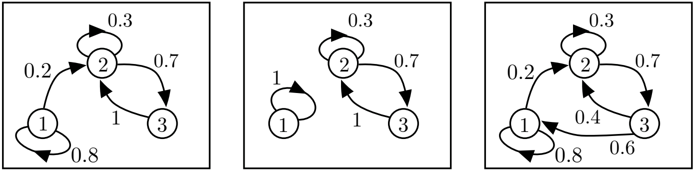

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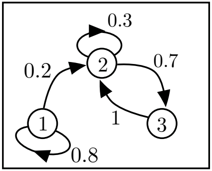

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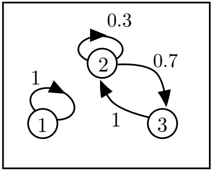

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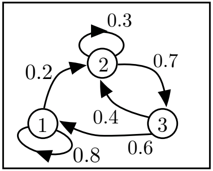

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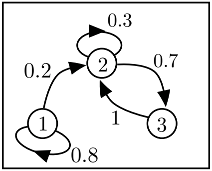

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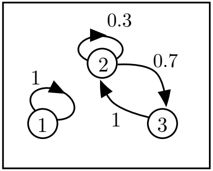

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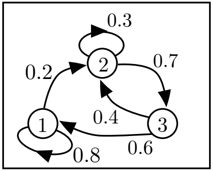

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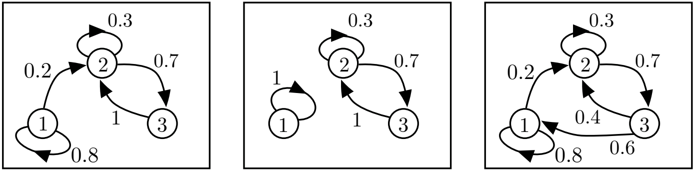

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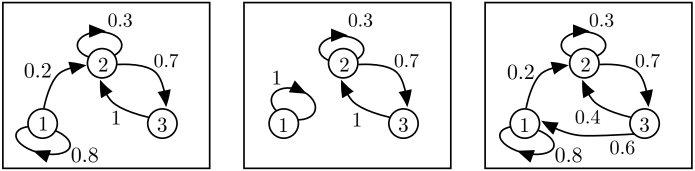

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  Only one stationary distribution if irreducible (or connected.)

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Proof: Lecture note 21 gives a plausibility argument.

□

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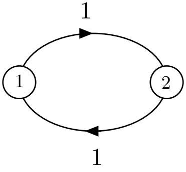

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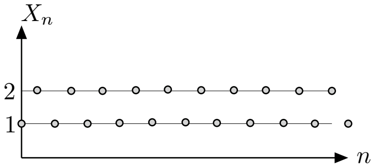

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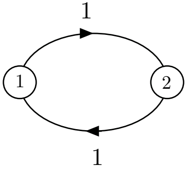

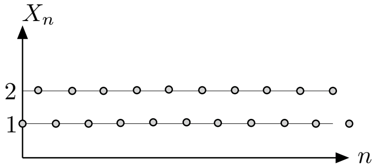

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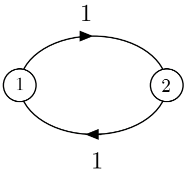

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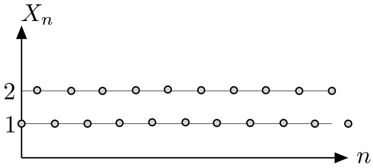

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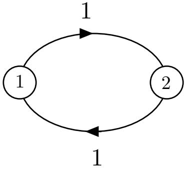

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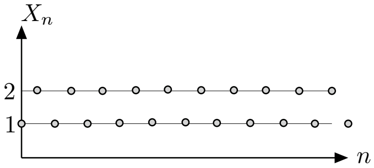

The fraction of time in state 1 converges to $1/2$, which is $\pi(1)$.

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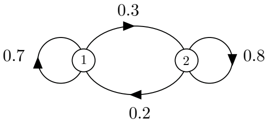

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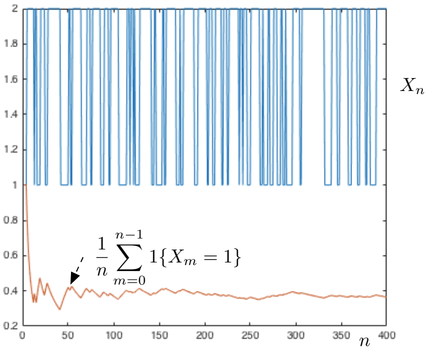

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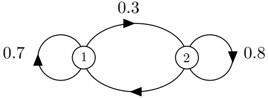

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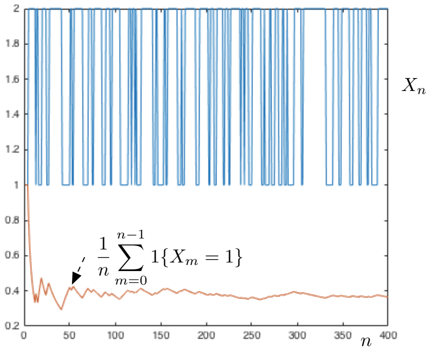

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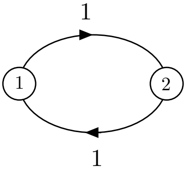

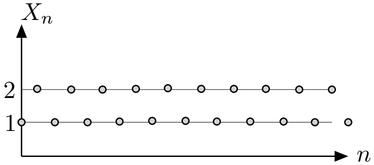

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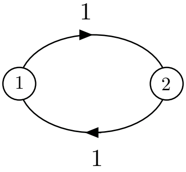

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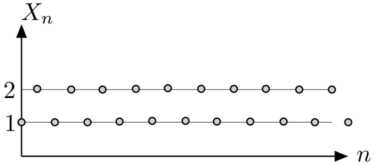

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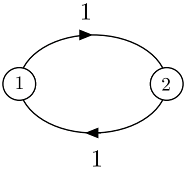

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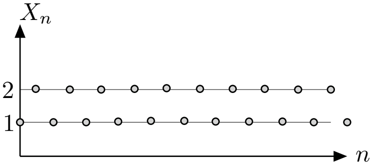

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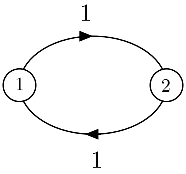

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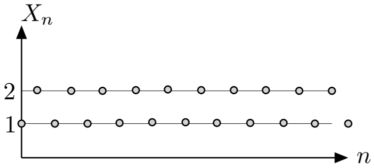

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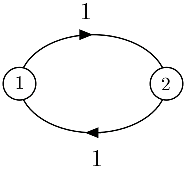

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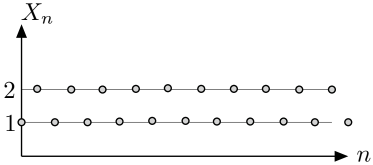

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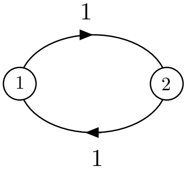

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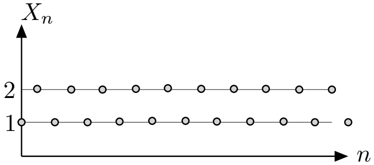

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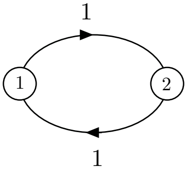

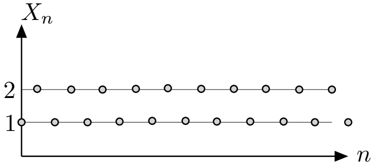

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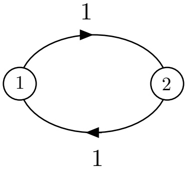

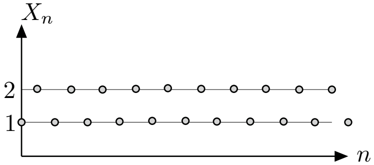

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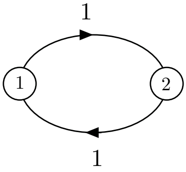

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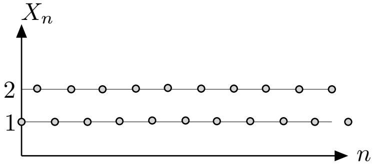

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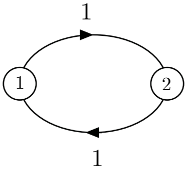

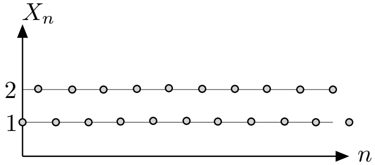

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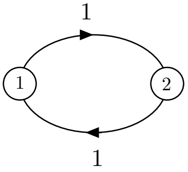

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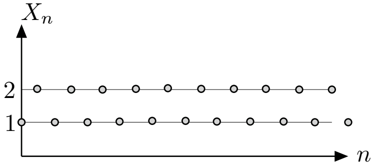

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Notice, all cycles or closed walks have even length.

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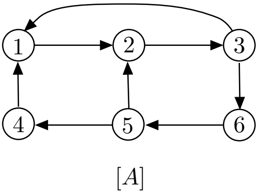

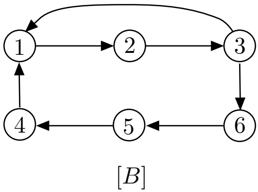

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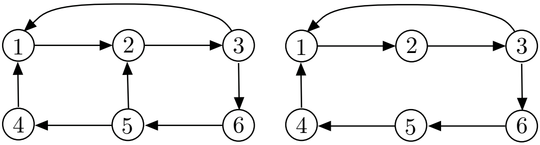

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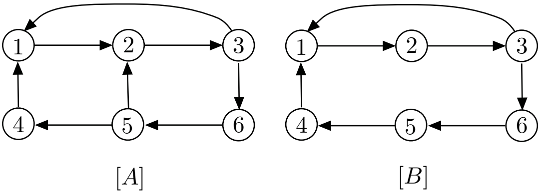

[A]: Closed walks of length 3 and length 4

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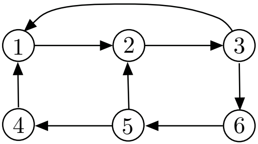

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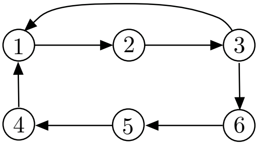

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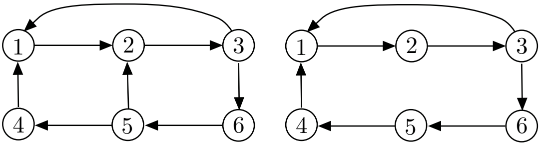

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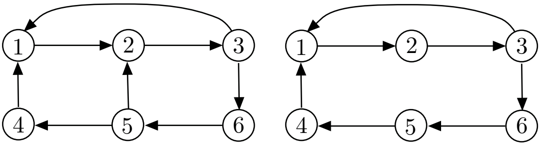

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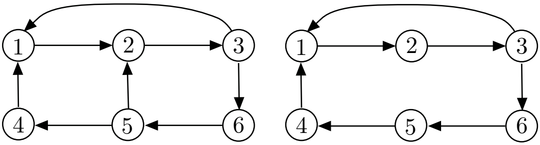
Periodicity

Definition: Periodicity is gcd of the lengths of all closed walks in irreducible chain. Previous example: 2.
Definition If periodicity is 1, Markov chain is said to be aperiodic. Otherwise, it is periodic.
Example



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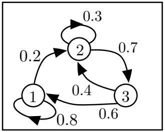

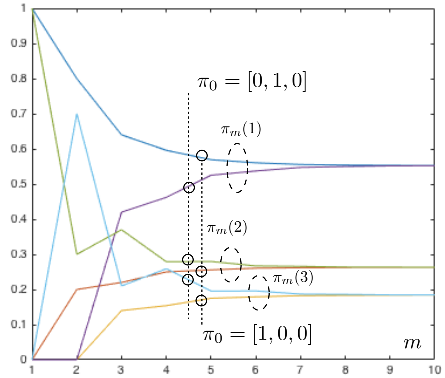

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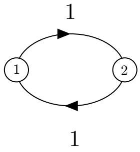
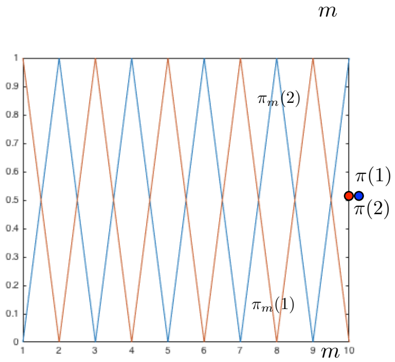


$\pi = [0.5, 0.5]$



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Surprisingly, people tend to be reinforced in their original belief, even when the evidence accumulates against it.

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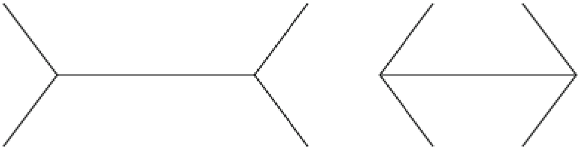

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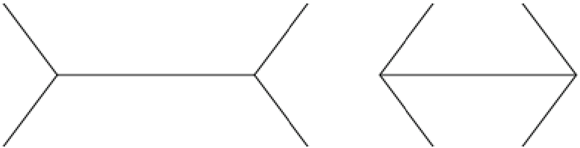

It is difficult to think clearly!

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