Today.

Comment: Add 0. Poll.

Add $(k-k)$.

Induction: Some quibbles.

What did you learn in 61A?

Induction and Recursion

Couple of more induction proofs.

Stable Marriage.

Some quibbles.

The induction principle works on the natural numbers.

Proves statements of form: $\forall n \in \mathbb{N}, P(n)$.

Yes.

What if the statement is only for $n \geq 3$?

$\forall n \in \mathbb{N}, (n \geq 3) \implies P(n)$

Restate as:
$\forall n \in \mathbb{N}, Q(n)$ where $Q(n) =$ "$(n \geq 3) \implies P(n)$".

Base Case: typically start at 3.
Since $\forall n \in \mathbb{N}, Q(n) \implies Q(n+1)$ is trivially true before 3.

Can you do induction over other things? Yes.

Any set where any subset of the set has a smallest element.

In some sense, the natural numbers.

Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

Instead of proof, let's write some code!

```
def find-x-y(n):
   if (n==12) return (3,0)
   elif (n==13): return(2,1)
   elif (n==14): return(1,2)
   elif (n==15): return(0,3)
   else:
      (x',y') = find-x-y(n-4)
      return(x'+1,y')
```

Base cases: P(12) , P(13) , P(14) , P(15). Yes.

Strong Induction step:
 Recursive call is correct: $P(n-4) \implies P(n)$.
 $n-4 = 4x' + 5y' \implies n = 4(x'+1) + 5(y')$

Slight differences: showed for all $n \geq 16$ that $\wedge_{i=4}^{n-1} P(i) \implies P(n)$.

Strengthening: need to...

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Base: $P(1)$. $1 \leq 2$.
Ind Step: $\sum_{i=1}^{k} \frac{1}{i^2} \leq 2$.

$\sum_{i=1}^{k+1} \frac{1}{i^2}$

$= \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2}$.

$\leq 2 + \frac{1}{(k+1)^2}$

Uh oh?

Hmmm... It better be that any sum is *strictly less than* 2.

How much less? At least by $\frac{1}{(k+1)^2}$ for S_k.

"$S_k \leq 2 - \frac{1}{(k+1)^2}$" $\implies$ "$S_{k+1} \leq 2$"
Induction step works! No! Not the same statement!!!!
Need to prove "$S_{k+1} \leq 2 - \frac{1}{(k+2)^2}$".

Darn!!!

Strenthening: how?

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n)$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Proof:

Ind hyp: $P(k)$ — "$S_k \leq 2 - f(k)$"

Prove: $P(k+1)$ – "$S_{k+1} \leq 2 - f(k+1)$"

$$\begin{aligned} S(k+1) &= S_k + \frac{1}{(k+1)^2} \\ &\leq 2 - f(k) + \frac{1}{(k+1)^2} \text{ By ind. hyp.} \end{aligned}$$

Choose $f(k+1) \leq f(k) - \frac{1}{(k+1)^2}$.

$$\implies S(k+1) \leq 2 - f(k+1).$$

Can you?

Subtracting off a quadratically decreasing function every time.

Maybe a linearly decreasing function to keep positive?

Try $f(k) = \frac{1}{k}$

$$\frac{1}{k+1} \leq \frac{1}{k} - \frac{1}{(k+1)^2}?$$

$$1 \leq \frac{k+1}{k} - \frac{1}{k+1} \quad \text{Multiplied by } k+1.$$

$$1 \leq 1 + \left(\frac{1}{k} - \frac{1}{k+1}\right) \quad \text{Some math. So yes!}$$

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$.

Stable Matching Problem

- n candidates and n jobs.
- Each job has a ranked preference list of candidates.
- Each candidate has a ranked preference list of jobs.

How should they be matched?

Count the ways..

- Maximize total satisfaction.
- Maximize number of first choices.
- Maximize worse off.
- Minimize difference between preference ranks.

The best laid plans..

Consider the pairs..

- (Anthony) Davis and Pelicans
- (Lonzo) Ball and Lakers

Davis prefers the Lakers.

Lakers prefer Davis.

Uh..oh. Sad Lonzo and Pelicans.

So..

Produce a pairing where there is no crazy moves!

Definition: A **pairing** is disjoint set of n job-candidate pairs.

Example: A pairing $S = \{(Lakers, Ball); (Pelicans, Davis)\}$.

Definition: A **rogue couple** b, g^* for a pairing S:
b and g^* prefer each other to their partners in S

Example: Davis and Lakers are a rogue couple in S.

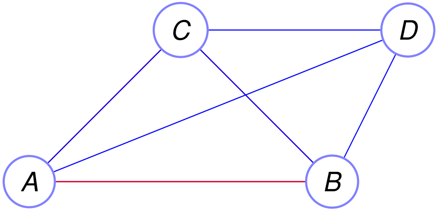
A stable pairing??

Given a set of preferences.

Is there a stable pairing?
How does one find it?

Consider a single type version: stable roommates.

A	B	C	D
B	C	A	D
C	A	B	D
D	A	B	C



The Propose and Reject Algorithm.

Each Day:

1. Each job **proposes** to its favorite candidate on its list.
2. Each candidate rejects all but their favorite proposer (whom they put on a string.)
3. Rejected job crosses rejecting candidate off its list.

Stop when each job gets exactly one proposal.
Does this terminate?

...produce a pairing?

....a stable pairing?

Do jobs or candidates do “better”?

Example.

Jobs			
A	~~1~~	2	3
B	~~1~~	~~2~~	3
C	~~2~~	1	3

Candidates			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A	~~A~~ , C	C	C
2	C	B, ~~C~~	B	A,~~B~~	A
3					B

Termination.

Every non-terminated day a job **crossed** an item off the list.

Total size of lists? n jobs, n length list. n^2

Terminates in $\leq n^2$ steps!

It gets better every day for candidates.

Improvement Lemma: It just gets better for candidates
If on day t a candidate g has a job b on a string,
any job, b', on candidate g's string for any day $t' > t$
is at least as good as b.

Example: Candidate "Alice" has job "Amalgamated Concrete" on string on day 5.

She has job "Amalgamated Asphalt" on string on day 7.

Does Alice prefer "Almalgamated Asphalt" or "Amalgamated Concrete"?

g - 'Alice', b - 'Am. Con.', b' - 'Am. Asph.', $t = 5$, $t' = 7$.

Improvement Lemma says she prefers 'Almalgamated Asphalt'.

Day 10: Can Alice have "Amalgamated Asphalt" on her string? Yes.

Alice prefers day 10 job as much as day 7 job. Here, $b = b'$.

Why is lemma true?

Proof Idea: She can always keep the previous job on the string.

Improvement Lemma

Improvement Lemma: It just gets better for candidates.

If on day t a candidate g has a job b on a string, any job, b', on g's string for any day $t' > t$ is at least as good as b.

Proof:
$P(k)$- - "job on g's string is at least as good as b on day $t+k$"

$P(0)$– true. Candidate has b on string.

Assume $P(k)$. Let b' be job **on string** on day $t+k$.

On day $t+k+1$, job b' comes back.
Candidate g can choose b', or do better with another job, b''

That is, $b' \leq b$ by induction hypothesis.
And b'' is better than b' by algorithm.
$\implies$ Candidate does at least as well as with b.

$P(k) \implies P(k+1)$.
And by principle of induction, lemma holds for every day after t. $\square$

Pairing when done.

Lemma: Every job is matched at end.
(Launch Proof poll.)

Proof:
If not, a job *b* must have been rejected *n* times.

Every candidate has been proposed to by *b*,
and Improvement lemma

$\implies$ each candidate has a job on a string.

and each job is on at most one string.

n candidates and *n* jobs. Same number of each.

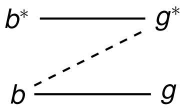
$\implies$ *b* must be on some candidate's string!

Contradiction. $\square$

Pairing is Stable.

Lemma: There is no rogue couple for the pairing formed by traditional marriage algorithm.

Proof:
Assume there is a rogue couple; (b, g^*)



b prefers g^* to g.

g^* prefers b to b^*.

Job b proposes to g^* before proposing to g.

So g^* rejected b (since he moved on)

By improvement lemma, g^* prefers b^* to b.

Contradiction! □

Good for jobs? candidates?

Is the Job-Proposes better for jobs? for candidates?

Definition: A **pairing is x-optimal** if $x's$ partner
is its best partner in any stable pairing.

Definition: A **pairing is x-pessimal** if $x's$ partner
is its worst partner in any stable pairing.

Definition: A **pairing is job optimal** if it is x-optimal for **all** jobs x.

..and so on for job pessimal, candidate optimal, candidate pessimal.

Claim: The optimal partner for a job must be first in its preference list.

True? False? False!

Subtlety here: Best partner in any stable pairing.
As well as you can be in a globally stable solution!

Question: Is there a job or candidate optimal pairing?
Is it possible:
b-optimal pairing different from the b'-optimal pairing!
Yes? No?

Understanding Optimality: by example.

A:	1,2	1:	A,B
B:	1,2	2:	B,A

Consider pairing: $(A,1),(B,2)$.

Stable? Yes.

Optimal for B?
Notice: only one stable pairing.
So this is the best B can do in a stable pairing.
So optimal for B.

Also optimal for A, 1 and 2. Also pessimal for A,B,1 and 2.

A:	1,2	1:	B,A
B:	2,1	2:	A,B

Pairing S: $(A,1),(B,2)$. Stable? Yes.

Pairing T: $(A,2),(B,1)$. Also Stable.

Which is optimal for A? S Which is optimal for B? S
Which is optimal for 1? T Which is optimal for 2? T

Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

Theorem: Job Propose and Reject produces a job-optimal pairing.

Proof:
Assume not: there is a job b does not get optimal candidate, g.

There is a stable pairing S where b and g are paired.

Let t be first day job b gets rejected
by its optimal candidate g who it is paired with
in stable pairing S.

b^* - knocks b off of g's string on day t $\implies$ g prefers b^* to b

By choice of t, b^* likes g at least as much as optimal candidate.

$\implies$ b^* prefers g to its partner g^* in S.

Rogue couple for S.
So S is not a stable pairing. Contradiction. $\square$

Notes: S - stable. $(b^*, g^*) \in S$. But (b^*, g) is rogue couple!

Used Well-Ordering principle...Induction.

How about for candidates?

Theorem: Job Propose and Reject produces candidate-pessimal pairing.

T – pairing produced by JPR.

S – worse stable pairing for candidate g.

In T, (g, b) is pair.

In S, (g, b^*) is pair.

g prefers b to b^*.

T is job optimal, so b prefers g to its partner in S.

(g, b) is Rogue couple for S

S is not stable.

Contradiction. $\square$

Notes: Not really induction.
Structural statement: Job optimality $\implies$ Candidate pessimality.

Quick Questions.

How does one make it better for candidates?

Propose and Reject - stable matching algorithm. One side proposes.

Jobs Propose $\implies$ job optimal.

Candidates propose. $\implies$ optimal for candidates.

Residency Matching..

The method was used to match residents to hospitals.

Hospital optimal....

..until 1990's...Resident optimal.

Another variation: couples.

Takeaways.

Analysis of cool algorithm with interesting goal: stability.

"Economic": different utilities.

Definition of optimality: best utility in stable world.

Action gives better results for individuals but gives instability.

Induction over steps of algorithm.

Proofs carefully use definition:
 Optimality proof:
  contradiction of the existence of a better pairing.