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Comment: Add 0.

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Add $(k-k)$.

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Induction: Some quibbles.

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What did you learn in 61A?

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Induction and Recursion

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Induction and Recursion

Couple of more induction proofs.

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Induction: Some quibbles.

What did you learn in 61A?

Induction and Recursion

Couple of more induction proofs.

Stable Marriage.

Some quibbles.

The induction principle works on the natural numbers.

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Proves statements of form: $\forall n \in \mathbb{N}, P(n)$.

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In some sense, the natural numbers.

Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

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Slight differences: showed for all $n \geq 16$ that $\wedge_{i=4}^{n-1} P(i) \implies P(n)$.

Strengthening: need to...

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

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Need to prove "$S_{k+1} \leq 2 - \frac{1}{(k+2)^2}$".

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"$S_k \leq 2 - \frac{1}{(k+1)^2}$" $\implies$ "$S_{k+1} \leq 2$"

Induction step works! No! Not the same statement!!!!

Need to prove "$S_{k+1} \leq 2 - \frac{1}{(k+2)^2}$".

Strengthening: need to...

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Base: $P(1)$. $1 \leq 2$.
Ind Step: $\sum_{i=1}^{k} \frac{1}{i^2} \leq 2$.

$$\sum_{i=1}^{k+1} \frac{1}{i^2}$$
$$= \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2}.$$
$$\leq 2 + \frac{1}{(k+1)^2}$$

Uh oh?

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Darn!!!

Strenthening: how?

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n)$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

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Ind hyp: $P(k)$

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Prove: $P(k+1)$ – "$S_{k+1} \leq 2 - f(k+1)$"

$$S(k+1) = S_k + \frac{1}{(k+1)^2}$$

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Maybe a linearly decreasing function to keep positive?

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Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$.

Stable Matching Problem

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- n candidates and n jobs.

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- Each job has a ranked preference list of candidates.

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How should they be matched?

Count the ways..

- Maximize total satisfaction.

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Count the ways..

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- Maximize number of first choices.
- Maximize worse off.
- Minimize difference between preference ranks.

The best laid plans..

Consider the pairs..

- (Anthony) Davis and Pelicans
- (Lonzo) Ball and Lakers

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Davis prefers the Lakers.

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Davis prefers the Lakers.

Lakers prefer Davis.

The best laid plans..

Consider the pairs..

- (Anthony) Davis and Pelicans
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Davis prefers the Lakers.

Lakers prefer Davis.

Uh..oh.

The best laid plans..

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Lakers prefer Davis.

Uh..oh. Sad Lonzo and Pelicans.

So..

Produce a pairing where there is no crazy moves!

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Definition: A **pairing** is disjoint set of *n* job-candidate pairs.

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Example: A pairing $S = \{(\mathit{Lakers}, \mathit{Ball}); (\mathit{Pelicans}, \mathit{Davis})\}$.

So..

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b and g^* prefer each other to their partners in S

Example: Davis and Lakers are a rogue couple in S.

A stable pairing??

Given a set of preferences.

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Is there a stable pairing?
How does one find it?

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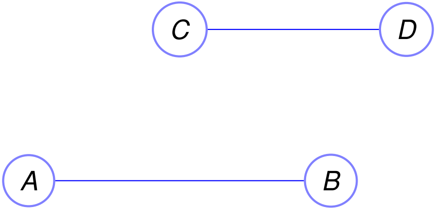
Consider a single type version: stable roommates.

A	B	C	D
B	C	A	D
C	A	B	D
D	A	B	C



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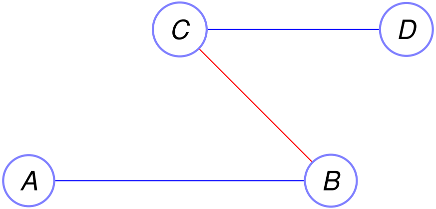

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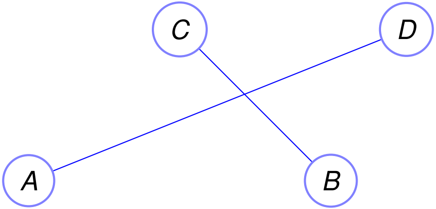

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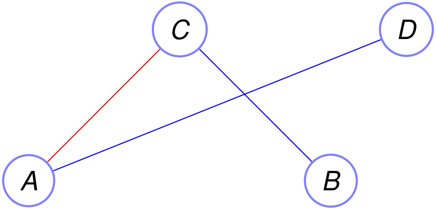

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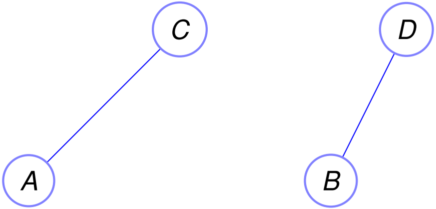

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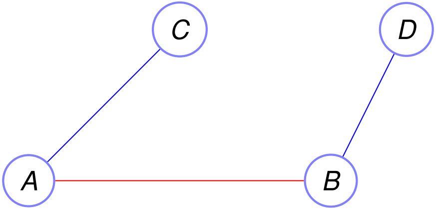

A stable pairing??

Given a set of preferences.

Is there a stable pairing?
How does one find it?

Consider a single type version: stable roommates.

A	B	C	D
B	C	A	D
C	A	B	D
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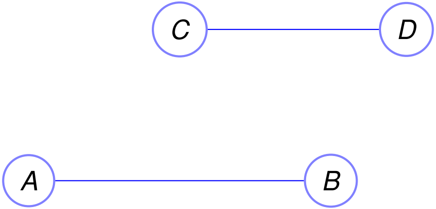

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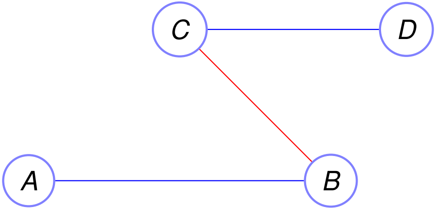

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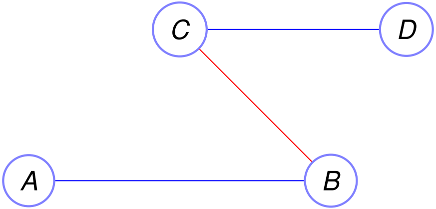

The Propose and Reject Algorithm.

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Each Day:

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Does this terminate?

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Do jobs or candidates do “better”?

Example.

Jobs			
A	1	2	3
B	1	2	3
C	2	1	3

Candidates			
1	C	A	B
2	A	B	C
3	A	C	B

Example.

Jobs			
A	1	2	3
B	1	2	3
C	2	1	3

Candidates			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1					
2					
3					

Example.

Jobs			
A	1	2	3
B	1	2	3
C	2	1	3

Candidates			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, B				
2	C				
3					

Example.

Jobs			
A	1	2	3
B	~~1~~	2	3
C	2	1	3

Candidates			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~				
2	C				
3					

Example.

Jobs			
A	1	2	3
B	~~1~~	2	3
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Candidates			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A			
2	C	B, C			
3					

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Jobs			
A	1	2	3
B	~~1~~	2	3
C	~~2~~	1	3

Candidates			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A			
2	C	B, ~~C~~			
3					

Example.

Jobs			
A	1	2	3
B	~~1~~	2	3
C	~~2~~	1	3

Candidates			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A	A , C		
2	C	B, ~~C~~	B		
3					

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Jobs			
A	~~1~~	2	3
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1	A, ~~B~~	A	~~A~~, C		
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3					

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A	~~1~~	2	3
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2	C	B, ~~C~~	B	A,B	
3					

Example.

Jobs			
A	~~1~~	2	3
B	~~1~~	~~2~~	3
C	~~2~~	1	3

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1	C	A	B
2	A	B	C
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	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A	~~A~~ , C	C	
2	C	B, ~~C~~	B	A, ~~B~~	
3					

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Jobs			
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2	C	B, ~~C~~	B	A, ~~B~~	A
3					B

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Jobs			
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	Day 1	Day 2	Day 3	Day 4	Day 5
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2	C	B, ~~C~~	B	A, ~~B~~	A
3					B

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Every non-terminated day a job **crossed** an item off the list.

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Terminates in $\leq n^2$ steps!

It gets better every day for candidates.

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Proof Idea: She can always keep the previous job on the string.

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Proof:
$P(k)$- - "job on g's string is at least as good as b on day $t+k$"

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On day $t+k+1$, job b' comes back.

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Candidate g can choose b',

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On day $t+k+1$, job b' comes back.
Candidate g can choose b', or do better with another job, b''

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$\Longrightarrow$ Candidate does at least as well as with b.

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$P(k) \implies P(k+1)$.

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And b'' is better than b' by algorithm.
$\implies$ Candidate does at least as well as with b.

$P(k) \implies P(k+1)$.
And by principle of induction, lemma holds for every day after t.

Improvement Lemma

Improvement Lemma: It just gets better for candidates.

If on day t a candidate g has a job b on a string, any job, b', on g's string for any day $t' > t$ is at least as good as b.

Proof:
$P(k)$- - "job on g's string is at least as good as b on day $t+k$"

$P(0)$– true. Candidate has b on string.

Assume $P(k)$. Let b' be job **on string** on day $t+k$.

On day $t+k+1$, job b' comes back.
Candidate g can choose b', or do better with another job, b''

That is, $b' \leq b$ by induction hypothesis.
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Lemma: Every job is matched at end.
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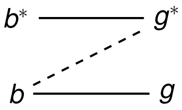

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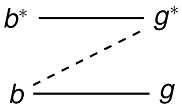

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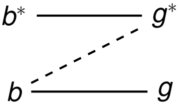

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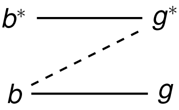

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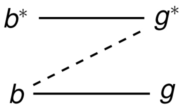

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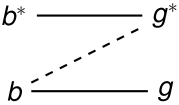

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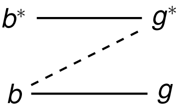

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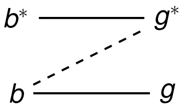

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Which is optimal for A?

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Pairing S: $(A,1),(B,2)$. Stable? Yes.

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Which is optimal for A? S

Understanding Optimality: by example.

A:	1,2	1:	A,B
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Consider pairing: $(A,1),(B,2)$.

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Job Propose and Candidate Reject is optimal!

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For jobs? For candidates?

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Quick Questions.

How does one make it better for candidates?

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Propose and Reject - stable matching algorithm. One side proposes.

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Jobs Propose $\implies$ job optimal.

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Candidates propose.

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Candidates propose. $\Longrightarrow$ optimal for candidates.

Residency Matching..

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The method was used to match residents to hospitals.

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Hospital optimal....

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Another variation: couples.

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Proofs carefully use definition:

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Proofs carefully use definition:
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