Lecture 7. Outline.

1. Isoperimetric inequality for hypercube.
2. Modular Arithmetic.
   Clock Math!!!
3. Inverses for Modular Arithmetic: Greatest Common Divisor.
   Division!!!
4. Euclid's GCD Algorithm.
   A little tricky here!

Isoperimetry.

For 3-space:

The sphere minimizes surface area to volume.

Surface Area: $4\pi r^2$, Volume: $\frac{4}{3}\pi r^3$.

Ratio: $1/3r = \Theta(V^{-1/3})$.

Graphical Analog: Cut into two pieces and find ratio of edges/vertices on small side.

Tree: $\Theta(1/|V|)$.

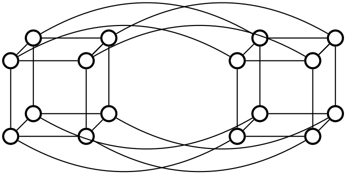
Hypercube: $\Theta(1)$.

Surface Area is roughly at least the volume!

Recursive Definition.

A 0-dimensional hypercube is a node labelled with the empty string of bits.

An n-dimensional hypercube consists of a 0-subcube (1-subcube) which is a $n-1$-dimensional hypercube with nodes labelled $0x$ ($1x$) with the additional edges $(0x, 1x)$.



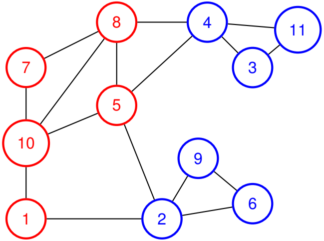
Hypercube: Can't cut me!

Thm: Any subset S of the hypercube where $|S| \leq |V|/2$ has $\geq |S|$ edges connecting it to $V - S$; $|E \cap S \times (V - S)| \geq |S|$

Terminology:
$(S, V - S)$ is cut.
$(E \cap S \times (V - S))$ - cut edges.

Restatement: for any cut in the hypercube, the number of cut edges is at least the size of the small side.

Cuts in graphs.



S is red, $V - S$ is blue.

What is size of cut?

Number of edges between red and blue. 4.

Hypercube: any cut that cuts off x nodes has $\geq x$ edges.

Proof of Large Cuts.

Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side.
Proof:
Base Case: $n = 1$ V= $\{0,1\}$.
$S = \{0\}$ has one edge leaving. $|S| = \phi$ has 0.

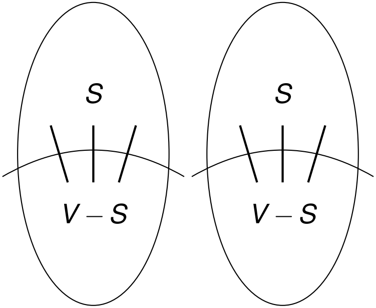
Induction Step Idea

Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side.

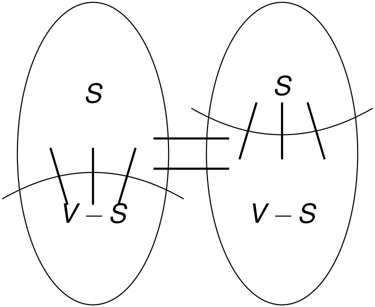
Use recursive definition into two subcubes.

Two cubes connected by edges.

Case 1: Count edges inside subcube inductively.



Case 2: Count inside and across.



Induction Step

Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

Proof: Induction Step.
Recursive definition:
$H_0 = (V_0, E_0)$,$H_1 = (V_1, E_1)$, edges E_x that connect them.
$H = (V_0 \cup V_1, E_0 \cup E_1 \cup E_x)$
$S = S_0 \cup S_1$ where S_0 in first, and S_1 in other.

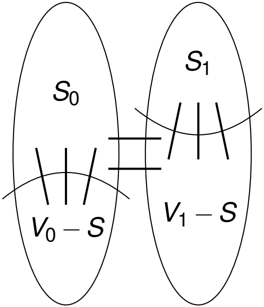
Case 1: $|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2$
Both S_0 and S_1 are small sides. So by induction.
Edges cut in $H_0 \geq |S_0|$.
Edges cut in $H_1 \geq |S_1|$.

Total cut edges $\geq |S_0| + |S_1| = |S|$. $\square$

Induction Step. Case 2.

Thm: For any cut $(S, V-S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

Proof: Induction Step. Case 2.



$|S_0| \geq |V_0|/2$.

Recall Case 1: $|S_0|, |S_1| \leq |V|/2$

$|S_1| \leq |V_1|/2$ since $|S| \leq |V|/2$.

$\implies \geq |S_1|$ edges cut in E_1.

$|S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2$

$\implies \geq |V_0| - |S_0|$ edges cut in E_0.

Edges in E_x connect corresponding nodes.

$\implies = |S_0| - |S_1|$ edges cut in E_x.

Total edges cut:

$\geq |S_1| + |V_0| - |S_0| + |S_0| - |S_1| = |V_0|$

$|V_0| = |V|/2 \geq |S|$. $\square$

Also, case 3 where $|S_1| \geq |V|/2$ is symmetric.

Hypercubes and Boolean Functions.

The cuts in the hypercubes are exactly the transitions from 0 sets to 1 set on boolean functions on $\{0,1\}^n$.

Central area of study in computer science!

Yes/No Computer Programs $\equiv$ Boolean function on $\{0,1\}^n$

Central object of study.

Hypercubes and Boolean Functions.

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Central object of study.

Modular Arithmetic.

Applications: cryptography, error correction.

Key idea for modular arithmetic.

Theorem: If $d|x$ and $d|y$, then $d|(y-x)$.

Proof:
$x = ad$, $y = bd$,
$(x-y) = (ad-bd) = d(a-b) \implies d|(x-y)$.

$\square$

Theorem: Every number $n \geq 2$ can be represented as a product of primes.

Proof: Either prime, or $n = a \times b$, and use strong induction. $\square$
(Uniqueness? Later.)

Next Up.

Modular Arithmetic.

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

What time is it in 5 hours? 6:00!

What time is it in 15 hours? 16:00!

Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.

Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.

$101 = 12 \times 8 + 5$.

5 is the same as 101 for a 12 hour clock system.

Clock time equivalent up to addition of any integer multiple of 12.

Custom is only to use the representative in $\{12, 1, \ldots, 11\}$

(Almost remainder, except for 12 and 0 are equivalent.)

Day of the week.

Today is Thursday.
What day is it a year from now? on September 17, 2021?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
5 days from now. day 9 or day 2 or Tuesday.
25 days from now. day 29 or day 1. $29 = (7)4 + 1$
two days are equivalent up to addition/subtraction of multiple of 7.
11 days from now is day 1 which is Monday!

What day is it a year from now?
Next year is not a leap year. So 365 days from now.
Day 4+365 or day 369.
Smallest representation:
subtract 7 until smaller than 7.
divide and get remainder.
369/7 leaves quotient of 52 and remainder 3. $369 = 7(52) + 5$
or September 18, 2020 is a Friday.

Years and years...

80 years from now? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
Remainder when dividing by 7? $104 = 14 \times 7 + 6$.
Or September 18, 2100 is Saturday!

Further Simplify Calculation:
20 has remainder 6 when divided by 7.
60 has remainder 4 when divided by 7.
Get Day: $2 + 2 \times 6 + 1 \times 4 = 18$.
Or Day 6. September 18, 2100 is Saturday.

"Reduce" at any time in calculation!

Modular Arithmetic: refresher.

x **is congruent to** y **modulo** m or "$x \equiv y \pmod{m}$"
if and only if $(x - y)$ is divisible by m.
...or x and y have the same remainder w.r.t. m.
...or $x = y + km$ for some integer k.

Mod 7 equivalence classes:
$\{\ldots, -7, 0, 7, 14, \ldots\}$ $\{\ldots, -6, 1, 8, 15, \ldots\}$ ...

Useful Fact: Addition, subtraction, multiplication can be done with any equivalent x and y.

or " $a \equiv c \pmod{m}$ and $b \equiv d \pmod{m}$
$\implies a + b \equiv c + d \pmod{m}$ and $a \cdot b = c \cdot d \pmod{m}$"

Proof: If $a \equiv c \pmod{m}$, then $a = c + km$ for some integer k.
If $b \equiv d \pmod{m}$, then $b = d + jm$ for some integer j.
Therefore, $a + b = c + d + (k + j)m$ and since $k + j$ is integer.
$\implies a + b \equiv c + d \pmod{m}$. $\square$

Can calculate with representative in $\{0, \ldots, m - 1\}$.

Notation

$x \pmod{m}$ or $\mod(x, m)$
- remainder of x divided by m in $\{0, \ldots, m-1\}$.

$\mod(x, m) = x - \lfloor \frac{x}{m} \rfloor m$

$\lfloor \frac{x}{m} \rfloor$ is quotient.

$\mod(29, 12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12 = 29 - (2) \times 12 = \cancel{4} = 5$

Work in this system.
$a \equiv b \pmod{m}$.
Says two integers a and b are equivalent modulo m.

Modulus is m

$6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}$.

$6 = 3 + 3 = 3 + 10 \pmod{7}$.

Generally, not $6 \pmod{7} = 13 \pmod{7}$.
But probably won't take off points, still hard for us to read.

Inverses and Factors.

Division: multiply by multiplicative inverse.

$$2x = 3 \implies (\frac{1}{2}) \cdot 2x = (\frac{1}{2}) \cdot 3 \implies x = \frac{3}{2}.$$

Multiplicative inverse of x is y where $xy = 1$;
1 **is multiplicative identity element.**

In modular arithmetic, 1 is the multiplicative identity element.

Multiplicative inverse of x **mod** m is y with $xy = 1 \pmod{m}$.

For 4 modulo 7 inverse is 2: $2 \cdot 4 \equiv 8 \equiv 1 \pmod{7}$.

Can solve $4x = 5 \pmod{7}$.

$2 \cdot 4x = 2 \cdot 5 \pmod{7}$

$8x = 10 \pmod{7}$

$x = 3 \pmod{7}$

Check! $4(3) = 12 = 5 \pmod{7}$.

$x = 3 \pmod{7}$ ::: Check! $4(3) = 12 = 5 \pmod{7}$.

For 8 modulo 12: no multiplicative inverse!

"Common factor of 4" $\implies$
$8k - 12\ell$ is a multiple of four for any ℓ and k $\implies$
$8k \not\equiv 1 \pmod{12}$ for any k.

Greatest Common Divisor and Inverses.

Thm:
If greatest common divisor of x and m, $\gcd(x,m)$, is 1, then x has a multiplicative inverse modulo m.

Proof $\implies$:
Claim: The set $S = \{0x, 1x, \ldots, (m-1)x\}$ contains $y \equiv 1 \mod m$ if all distinct modulo m.

Each of m numbers in S correspond to different one of m equivalence classes modulo m.

$\implies$ One must correspond to 1 modulo m. Inverse Exists!

Proof of Claim: If not distinct, then $\exists a, b \in \{0, \ldots, m-1\}$, $a \neq b$, where

$(ax \equiv bx \pmod{m}) \implies (a-b)x \equiv 0 \pmod{m}$

Or $(a-b)x = km$ for some integer k.

$gcd(x,m) = 1$

$\implies$ Prime factorization of m and x do not contain common primes.

$\implies$ $(a-b)$ factorization contains all primes in m's factorization.

So $(a-b)$ has to be multiple of m.

$\implies$ $(a-b) \geq m$. But $a, b \in \{0, \ldots m-1\}$. Contradiction. $\square$

Proof review. Consequence.

Thm: If $\gcd(x,m)=1$, then x has a multiplicative inverse modulo m.

Proof Sketch: The set $S=\{0x,1x,\ldots,(m-1)x\}$ contains $y\equiv 1 \mod m$ if all distinct modulo m.

... $\square$

For $x=4$ and $m=6$. All products of 4...
$S=\{0(4),1(4),2(4),3(4),4(4),5(4)\}=\{0,4,8,12,16,20\}$
reducing $\pmod 6$
$S=\{0,4,2,0,4,2\}$
Not distinct. Common factor 2. Can't be 1. No inverse.

For $x=5$ and $m=6$.
$S=\{0(5),1(5),2(5),3(5),4(5),5(5)\}=\{0,5,4,3,2,1\}$
All distinct, contains 1! 5 is multiplicative inverse of 5 $\pmod 6$.
(Hmm. What normal number is it own multiplicative inverse?) 1 -1.

$5x=3 \pmod 6$ What is x? Multiply both sides by 5.
$x=15=3 \pmod 6$

$4x=3 \pmod 6$ No solutions. Can't get an odd.
$4x=2 \pmod 6$ Two solutions! $x=2,5 \pmod 6$

Very different for elements with inverses.

Proof Review 2: Bijections.

If gcd(x,m) = 1.
Then the function $f(a) = xa \mod m$ is a bijection.
One to one: there is a unique pre-image.
Onto: the sizes of the domain and co-domain are the same.
$x = 3, m = 4$.
$f(1) = 3(1) = 3 \pmod 4, f(2) = 6 = 2 \pmod 4, f(3) = 1 \pmod 3$.
Oh yeah. $f(0) = 0$.

Bijection $\equiv$ unique pre-image and same size.
All the images are distinct. $\implies$ unique pre-image for any image.

$x = 2, m = 4$.
$f(1) = 2, f(2) = 0, f(3) = 2$
Oh yeah. $f(0) = 0$.

Not a bijection.

Finding inverses.

How to find the inverse?

How to find **if** x has an inverse modulo m?

Find gcd (x, m).
  Greater than 1? No multiplicative inverse.
  Equal to 1? Mutliplicative inverse.

Algorithm: Try all numbers up to x to see if it divides both x and m.

Very slow.

Inverses

Next up.

Euclid's Algorithm.
Runtime.
Euclid's Extended Algorithm.

Refresh

Does 2 have an inverse mod 8? No.
  Any multiple of 2 is 2 away from $0 + 8k$ for any $k \in \mathbb{N}$.

Does 2 have an inverse mod 9? Yes. 5
  $2(5) = 10 = 1 \mod 9$.

Does 6 have an inverse mod 9? No.
  Any multiple of 6 is 3 away from $0 + 9k$ for any $k \in \mathbb{N}$.
    $3 = gcd(6,9)$!

x has an inverse modulo m if and only if
  $gcd(x,m) > 1$? No.
  $gcd(x,m) = 1$? Yes.

Now what?:
  Compute gcd!
  Compute Inverse modulo m.

Divisibility...

Notation: $d|x$ means "d divides x" or
$x = kd$ for some integer k.

Fact: If $d|x$ and $d|y$ then $d|(x+y)$ and $d|(x-y)$.

Is it a fact? Yes? No?

Proof: $d|x$ and $d|y$ or
$x = \ell d$ and $y = kd$

$\implies x - y = kd - \ell d = (k-\ell)d \implies d|(x-y)$

$\square$

More divisibility

Notation: $d|x$ means "d divides x" or
$x = kd$ for some integer k.

Lemma 1: If $d|x$ and $d|y$ then $d|y$ and $d| \mod(x,y)$.

Proof:

$$\begin{aligned} \mod(x,y) &= x - \lfloor x/y \rfloor \cdot y \\ &= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s \\ &= kd - s\ell d \quad \text{for integers } k,\ell \text{ where } x = kd \text{ and } y = \ell d \\ &= (k - s\ell)d \end{aligned}$$

Therefore $d| \mod(x,y)$. And $d|y$ since it is in condition. $\square$

Lemma 2: If $d|y$ and $d| \mod(x,y)$ then $d|y$ and $d|x$.
Proof...: Similar. Try this at home. $\square$ish.

GCD Mod Corollary: $\gcd(x,y) = \gcd(y, \mod(x,y))$.
Proof: x and y have **same** set of common divisors as x and $\mod(x,y)$ by Lemma 1 and 2.
Same common divisors $\implies$ largest is the same. $\square$

Euclid's algorithm.

GCD Mod Corollary: $\gcd(x,y) = \gcd(y, \mod(x,y))$.

Hey, what's $\gcd(7,0)$? 7 since 7 divides 7 and 7 divides 0
What's $\gcd(x,0)$? x

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(define (euclid x y)
  (if (= y 0)
    x
    (euclid  y (mod x y))))   ***
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Theorem: (euclid x y) $= \gcd(x,y)$ if $x \geq y$.

Proof: Use Strong Induction.
Base Case: $y = 0$, "x divides y and x"
$\implies$ "x is common divisor and clearly largest."
Induction Step: $\mod(x,y) < y \leq x$ when $x \geq y$

call in line (***) meets conditions plus arguments "smaller"
and by strong induction hypothesis
computes $\gcd(y, \mod(x,y))$
which is $\gcd(x,y)$ by GCD Mod Corollary. $\square$

Modular Arithmetic Lecture in a minute.

Modular Arithmetic: $x \equiv y \pmod{N}$ if $x = y + kN$ for some integer k.

For $a \equiv b \pmod{N}$, and $c \equiv d \pmod{N}$,
$ac = bd \pmod{N}$ and $a + b = c + d \pmod{N}$.

Division? Multiply by multiplicative inverse.
$a \pmod{N}$ has multiplicative inverse, $a^{-1} \pmod{N}$.
If and only if $gcd(a,N) = 1$.

Why? If: $f(x) = ax \pmod{N}$ is a bijection on $\{1,\ldots,N-1\}$.
$ax - ay = 0 \pmod{N} \implies a(x-y)$ is a multiple of N.
If $gcd(a,N) = 1$,
then $(x-y)$ must contain all primes in prime factorization of N,
and is therefore be bigger than N.
Only if: For $a = xd$ and $N = yd$,
any $ma + kN = d(mx - ky)$ or is a multiple of d,
and is not 1.